XV. Great Picard, little Picard, and introduction to complex dynamics

§1 Picard's theorems

recall (Montel) $\mathcal{F}$: family of meromorphic functions on Ω
if $\mathcal{F}$ omits three values, then it must be normal

thm (Great Picard). f: meromorphic on $\{z \in \mathbb{C} \mid 0 < |z - z_0| < \delta\}$ some punctured disk
if f omits three values, then, f extends meromorphically to z_0

pf: Without loss of generality, we may assume that f omits $\{0, 1, \infty\}$
and the domain is $D^{\times} = D \setminus \{0\}$ no importance

We can zoom in near 0 to construct a family of funtion on $D^{\times}$

$$\mathcal{G} = \{ f_{\varepsilon}(z) = f(\varepsilon z) \mid 0 < \varepsilon < 1 \} \qquad 0 < \varepsilon z < 1 \Leftrightarrow 0 < z < \varepsilon^{-1}$$

By Montel, $\mathcal{G}$ must be normal. $\Rightarrow \exists\, f_{\varepsilon_n} \to g \quad (\varepsilon_n \to 0)$

g: meromorphic on $D^{\times}$

Since f_{ε_n} are all analytic, g is either analytic, or $g \equiv \infty$

case 1. g is analytic, let $M = \max\limits_{|\xi| = \frac{1}{2}} |g(\xi)|$

$\Rightarrow |f_{\varepsilon_n}(\xi)| \le 2M \quad \forall\, |\xi| = \frac{1}{2}$ and $n >> 1$.

$\quad \| $

$|f(\xi)| \quad |\xi| = \frac{\varepsilon_n}{2}$, let $\tilde{M} = \max\limits_{|\xi| = \frac{1}{2}} |f(\xi)|$

By the maximum principle, $|f(z)| \le 2M + \tilde{M} \quad \forall\, z$ with $\frac{\varepsilon_n}{2} < |z| < \frac{1}{2}$

But $\varepsilon_n \to 0 \Rightarrow z = 0$ is a removable singularity.

case 2. $g \equiv 0$. consider $\frac{1}{f}, \frac{1}{g}$. similar argument $\Rightarrow 0$ is a pole of f ※

thm (little Picard) any entire function can omit at most one value

pf: $f(z)$: entire if $f(z)$ omits two values
$\Rightarrow f(\frac{1}{w})$ omits three values $(\dots \cup \{\infty\})$
$\Rightarrow f(\frac{1}{w})$ extends to $w = 0$ meromorphically
$\Rightarrow$ $\boxed{\text{previous HW}}$ $f(z)$ must be a polynomial, assume every value of $\mathbb{C}$ ※

§2 Iteration of analytic / meromorphic function

application of normal family : understand the basics of complex dynamics
i.e. behavior of iteration of meromorphic functions

1° $f(z)$: polynomial or rational function

Consider $f(z): \hat{\mathbb{C}} \to \hat{\mathbb{C}} \longrightarrow$ iterates $f(f(\cdots f(z))\cdots)$
denote it by $f^n(z)$ (might be confusing)

e.g. $f(z) = z + \lambda \rightsquigarrow f^n(z) = z + n\lambda$
$f(z) = z^k \quad k \in \mathbb{N} \rightsquigarrow f^n(z) = z^{k^n}$

in general, NO closed formula

2° There are some equivalence here, given by change of coordinate (or automorphism)

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \\ \varphi\downarrow & & \downarrow\varphi \\ \hat{\mathbb{C}} & \xrightarrow[g]{} & \hat{\mathbb{C}} \end{array}$$

$\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, automorphism

$\Rightarrow$ $\boxed{\text{previous HW}}$ $\varphi(z) =$ Möbius transform

$\varphi(f(z)) = g(\varphi(z))$
$\Leftrightarrow g = \varphi \circ f \circ \varphi^{-1}$

e.g. For a degree d polynomial, we can conjugate it to a degree d polynomial with leading coefficient 1.

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0 \qquad a_d \neq 0$$

Consider $\varphi(z) = cz$

$$\varphi \circ f \circ \varphi^{-1} = \varphi\left(a_d \frac{z^d}{c^d} + a_{d-1}\frac{z^{d-1}}{c^{d-1}} + \cdots + a_0\right)$$
$$= a_d \frac{z^d}{c^{d-1}} + a_{d-1}\frac{z^{d-1}}{c^{d-2}} + \cdots + \frac{a_0}{c}$$

Take $c = (a_d)^{\frac{1}{d-1}}$ ※

More generally

$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ \varphi\downarrow & & \downarrow\varphi \\ V & \xrightarrow{g} & V \end{array}$$

f, g: analytic on U & V, respectively
φ: conformal map between U & V

3° (example study) $z \mapsto z^k \rightsquigarrow f^n(z) = z^{k^n}$
if $|z| > 1$ $f^n(z) \to \infty$, if $|z| < 1$ $f^n(z) \to 0$
if $|z| = 1$ behavior of $\{f^n\}$ near z is not so simple

<u>defn</u> f: rational function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$

The Fatou set of f, $\mathcal{F} = \{ z_0 \in \hat{\mathbb{C}} \mid \exists\, \mathcal{U}$: nbd of z_0 such that $\{f^n\}$ is normal on $\mathcal{U}\}$

Hence, the Fatou set is open

The Julia set is the complement of the Fatou set $\mathcal{J} = \hat{\mathbb{C}} \setminus \mathcal{F}$ and thus is closed.

4° (basic invariant property)

<u>thm</u> The Fatou and Julia sets are invariant under f.
Namely, $f(\mathcal{F}) = \mathcal{F}$ and $f(\mathcal{J}) = \mathcal{J}$

pf: We are only interested in the case when f is not a constant function.

Since $\mathcal{J} = \hat{\mathbb{C}} \setminus \mathcal{F}$, it is equivalent to $f(\mathcal{F}) \subset \mathcal{F}$ and $f^{-1}(\mathcal{F}) \subset \mathcal{F}$

[for $f^{-1}(\mathcal{F}) \subset \mathcal{F}$] $\underset{\underset{f^{-1}(z_0)}{\in}}{w_0} \xrightarrow{f} z_0 \xrightarrow{\{f^n\}} \mathbb{C}$

f^n: converges normally on a nbd of z_0
$\Rightarrow f^n \circ f = f^{n+1}$ converges normally on a nbd of w_0.

[for $f(\mathcal{F}) \subset \mathcal{F}$] $w_0 \longmapsto f(w_0)$

By open mapping, ... $\boxed{\text{HW}}$

5° [the polynomial case] Suppose $f(z) = z^{d \geq 2} +$ (lower order term)

(degree 1 is uninteresting, and we can conjugate so that the leading coefficient is 1)

Note that ∞ is a "stable" point, $f^n(z) \to \infty$ as $n \to \infty$ provided $|z| >> 1$

Examine it more quantitatively: $\exists\, R > 0$ such that $|f(z)| > 2|z|$ for $|z| > R$

$\Rightarrow |f^n(z)| > 2^n |z|$

It follows that $\{z \mid |z| > R\} \subset \mathcal{F}$ the Fatou set

<u>defn</u> $A(\infty)$: basin of attraction of $f(z)$: polynomial

is defined to be $\{z \in \hat{\mathbb{C}} \mid f^n(z) \to \infty \text{ as } n \to \infty\}$

Note that $A(\infty)$ is an open set containing ∞.

$$\left[\begin{aligned} z \in A(\infty) &\Rightarrow f^n(z) \in \{z \mid |z| > R\} \text{ for some } n \in \mathbb{N} \\ &\Rightarrow z \in (f^n)^{-1}(\{z \mid |z| > R\}) \Rightarrow A(\infty) = \bigcup_{n \geq 1} (f^n)^{-1}(\{z \mid |z| > R\}) \end{aligned}\right]$$

<u>thm</u> $f(z)$ = polynomial of deg ≥ 2 (may assume monic)

Then, $A(\infty)$ is open and connected. The Julia set $\mathcal{J}$ is its boundary.

$\mathcal{J} = \partial A(\infty)$, which is closed and bounded (hence compact)

Each bounded component in $\hat{\mathbb{C}} \setminus \mathcal{J}$ is simply-connected.

pf: • (connectedness of $A(\infty)$) It follows from the definition/construction

that $f(A(\infty)) \subset A(\infty)$ and $f^{-1}(A(\infty)) = A(\infty)$

Hence, $f(\partial A(\infty)) \cap A(\infty) = \phi$, and $f(\partial A(\infty)) \subset \partial A(\infty)$

$\Rightarrow f^n(\partial A(\infty)) \subset \partial A(\infty) \Rightarrow \{f^n\}$ is bdd on $\partial A(\infty)$

If $\mathcal{U} \subset \hat{\mathbb{C}} \setminus \partial A(\infty)$ a component not containing ∞.

(f: analytic $\Rightarrow$ analytic on $\mathcal{U}$. $\Rightarrow$ maximum principle)

then, $\max_{\mathcal{U}} |f^n| \leq \max_{\partial A(\infty)} |f^n| \Rightarrow \{f^n\}$ is bdd on $\mathcal{U} \Rightarrow \mathcal{U} \cap A(\infty) = \phi$

$\Rightarrow A(\infty)$ is connected.

• ($\mathcal{J} = \partial A(\infty)$) So far we have known (together with Montel) that $\hat{\mathbb{C}} \setminus \partial A(\infty) \subset \mathcal{F}$

Thus, $\mathcal{J} \subset \partial A(\infty)$

Given $z_0 \in \partial A(\infty)$, $\{f^n(z_0)\}$ is bounded

But for any nbd V of z_0, $V \cap A(\infty) \neq \phi$, and

$f^n(w) \to \infty$ as $n \to \infty$ $\forall w \in V \cap A(\infty)$

Thus, z_0 cannot belong to the Fatou set, $z_0 \in \mathcal{J}$

• (simply-connectedness) (definition of Ahlfors
Gamelin says more on the topology) ※

6° [example study : $f(z) = z^2 + c$)

$c = 0 \rightsquigarrow \mathcal{J} =$ unit circle

$c = -2$: $f(z) = z^2 - 2$ trick $\varphi(z) = z + \frac{1}{z}$ to cancel the constant term

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{g = z^2} & \mathcal{U} \\ \varphi \downarrow & & \downarrow \varphi \\ V & \xrightarrow{f} & V \end{array}$$

$f \circ \varphi = (z + \frac{1}{z})^2 - 2 = z^2 + \frac{1}{z^2}$

$= \varphi(z^2)$

By $\boxed{\text{previous HW}}$ $\mathcal{U} = \hat{\mathbb{C}} \setminus \bar{D}$, $V = \hat{\mathbb{C}} \setminus [-2, 2]$

$\downarrow g^n$

∞

Thus, $\hat{\mathbb{C}} \setminus [-2, 2] \subset A(\infty) \subset \mathcal{F}$ of $f(z) = z^2 - 2$

If $x \in [-2, 2]$ $f(x) = x^2 - 2 \in [-2, 2]$ $\Rightarrow$ $[-2, 2] = \partial A(\infty) = \mathcal{J}$

What about other values of c?

$f(z)$: polynomial, $\deg f \geq 2$, monic

$$A(\infty) = \bigcup_{n\geq 1} (f^n)^{-1}(\{z \mid |z|>R\}) \quad \text{and} \quad \mathcal{J} = \partial A(\infty)$$

one method to visualize $\mathcal{J}$

7° [more on the Julia set]

thm $f(z)$ as above. Given any $z_0 \in \mathcal{J}$ and open neighborhood $\mathcal{U}$ of z_0
$\exists N \in \mathbb{N}$ such that $\mathcal{J} \subset \mathcal{U} \cup f(\mathcal{U}) \cup \cdots \cup f^N(\mathcal{U})$
Moreover, $\bigcup_{k\geq 1} (f^k)^{-1}(z_1)$ is dense in $\mathcal{J}$ for any $z_1 \in \mathcal{J}$

reverse the order

rmk $z_1 \rightsquigarrow (f)^{-1}(z_1) \rightsquigarrow \cdots \rightsquigarrow (f^N)^{-1}(z_1)$ another method to visualize $\mathcal{J}$
see Gamelin for more

pf: By the definition of Julia set

$\mathcal{F} = \{f^n\}_{n\geq 1}$: not normal on $\mathcal{U} \ni z_0$

$\mathcal{F}$ already omits ∞ $\underset{\text{Montel}}{\Longrightarrow}$ $\mathcal{F}$ can omit at most one value in $\mathbb{C}$

case 1) $\bigcup_{n\geq 1} f^n(\mathcal{U}) = \mathbb{C} \Rightarrow \mathcal{J} \subset \bigcup_{n\geq 1} f^n(\mathcal{U})$
But $\mathcal{J}$ is compact, $\mathcal{J} \subset \bigcup_{n\geq 1}^{N} f^n(\mathcal{U})$

case 2) if $\bigcup_{n\geq 1} f^n(\mathcal{U}) = \mathbb{C} \setminus \{w_0\}$
$\Rightarrow$ $f^{-1}(w_0)$ consists of only w_0
otherwise $z_0 \neq w_0 \xrightarrow{f} z_0 \Rightarrow w_0 \in \bigcup_{n\geq 1} f^n(\mathcal{U})$
$z_0 \in \bigcup_{n\geq 1} f^n(\mathcal{U})$
$\Rightarrow f(z) = w_0 + (z-w_0)^n \Rightarrow \{f^n(z)\}$ near $w_0 \to w_0$
$\Rightarrow w_0 \in \mathcal{F}$, same as case 1).

Say it differently. given any open set $\mathcal{U} \subset \mathcal{J}$, and $z_1 \in \mathcal{J}$.
$\exists n \in \mathbb{N}$ so that $z_1 \in f^n(\mathcal{U}) \Rightarrow (f^n)^{-1}(z_1) \cap \mathcal{U} \neq \phi$

In other words, $\bigcup_{n\geq 1} (f^n)^{-1}(z_1)$ is dense in $\mathcal{J}$ ※

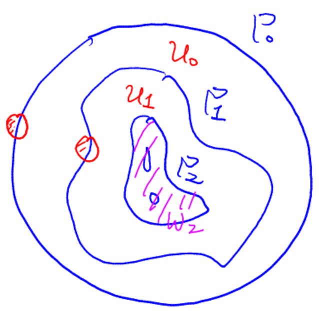
§3 connectedness of Julia sets

For $f(z) = z^{d\geq 2} + (\text{lower order terms})$, $A(\infty) = \bigcup_{n\geq 1} (f^n)^{-1}(\{z \mid |z| \geq \rho\})$

Examine the union carefully:

$\Gamma_0 = \{z \in \mathbb{C} \mid |z| = R\}$ R: large enough such that

$$\begin{cases} |f(z)| > R & \text{for } |z| \geq R \\ \Gamma_1 = f^{-1}(\Gamma_0) & \text{is "almost a circle" of radius } R^{\frac{1}{d}} \underset{d\text{-to-}1}{\xrightarrow{f}} \Gamma_0 \end{cases}$$



$\mathcal{U}_0$ = region sandwiches between Γ_0 & Γ_1

[$z \in \mathcal{U}_0 \Rightarrow |z| < R$
but $|f(z)| \geq R$ ⇜ minimum principle for f on $\mathcal{U}_0$]

Set $\Gamma_k = (f^k)^{-1}(\Gamma_0)$ $\mathcal{U}_k = (f^k)^{-1}(\mathcal{U}_0)$

If $z \in A(\infty)$, either $|z| > R$

or $\exists\, k \geq 0$ such that $|f^{k+1}(z)| > R$ but $|f^{k}(z)| \leq R$ $\iff f^{k}(z) \in \mathcal{U}_0 \cup \Gamma_0$ $\iff z \in \mathcal{U}_k \cup \Gamma_k$

assume $f \neq 0$ on $\mathcal{U}_0$ by minimum principle

In other words, $A(\infty)$ is the union of the disjoint sets
$\{z \mid |z| > R\}$, Γ_0, $\mathcal{U}_0$, Γ_1, $\mathcal{U}_1$ $\cdots\cdots$

Since $\mathrm{Crit}_f = \{ z \in \mathbb{C} \mid f'(z) = 0 \}$ are finite points. we may assume

$\Gamma_k \cap \mathrm{Crit}_f = \phi \quad \forall k \quad \Rightarrow f$ is conformal at each point of Γ_k

$\Rightarrow f$ map Γ_k to Γ_0 in a d^k-to-1 manner

(each point has the same multiplicity d_k)

Γ_k : finite union of disjoint, simple closed analytic curves
(inverse function theorem, closed & bounded $\Rightarrow$ compact)

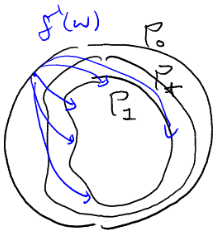
$\xRightarrow[\text{argument}]{\text{Some}}$
$\left\{ \begin{array}{l} \partial \mathcal{U}_k = \Gamma_k \cup \Gamma_{k+1} \\ \text{Let } W_k = (f^k)^{-1}(\{ z \in \mathbb{C} \mid |z| < R \}) \text{ is } \{ z \in \mathbb{C} \mid |z| < R \} \setminus \mathcal{U}_0 \cup \Gamma_0 \cup \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{k-1} \cup \Gamma_k \\ \text{and } \partial W_k = \Gamma_k \qquad \text{bdry } \Gamma_k \\ \text{Since } \mathbb{C} \setminus W_k \text{ is connected, each component of } \mathbb{C} \setminus W_k \text{ is conformal to } \mathbb{D} \\ \text{Also, } \mathbb{C} \setminus W_k \subset A(\infty) \Rightarrow W_k \supset \mathcal{J} \text{ and bounded components of } \mathcal{F} \end{array} \right.$

$\mathcal{J} = \lim_{k \to \infty} \Gamma_k = \lim_{k \to \infty} \mathcal{U}_k$

thm $\mathcal{J}$ is connected if and only if $\{ f^n(z) \}_{n \geq 1}$ is bounded $\forall z \in \mathrm{Crit}_f$

pf: [$\Leftarrow$] First of all, $\mathrm{Crit}_f \cap A(\infty) = \phi$

For each $w \in A(\infty)$, $f^{-1}(w)$ consists of d distinct points
and f^{-1} are actually d distinct analytic functions near w.



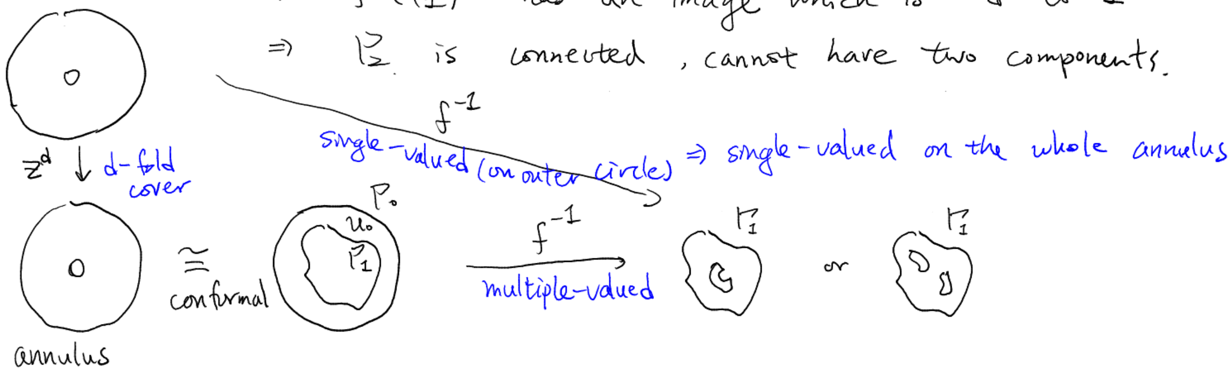
Consider $f^{-1}(w)$ for $w \in \Gamma_0$
travel along $w \in \Gamma_0$ d-times
$\iff$ (analytic continuation of) $f^{-1}(w)$ travel Γ_1 once.

Consider the analytic continuation of $f^{-1}(w)$ on Γ_t

$\Rightarrow$ must travel Γ_t d-times to come back $\quad \forall t \in [0, 1]$

$\Rightarrow f^{-1}(\Gamma_1)$ has an image which is d-to-1

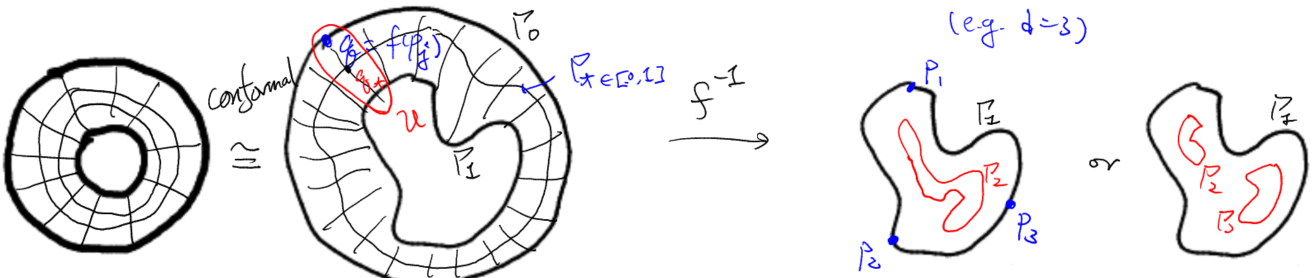
$\Rightarrow \Gamma_2$ is connected, cannot have two components.



Then $\mathcal{J} = \lim_{k \to \infty} \Gamma_k \qquad \Gamma_k$: connected and compact. $\mathcal{J}$: compact

$\boxed{\text{Exercise}}$ topology $\Rightarrow$ J must be connected

<u>another</u> explaination on the connectedness of Γ_2



(Since no critical points in $A(\infty)$) at each p. $\exists\, d$ distinct locally defined f^{-1}, each one is a conformal map.

(For each $p_j \in f^{-1}(q)$ $\rightsquigarrow$ d distinct f^{-1})

Define $\Gamma_1 \xrightarrow{\mu_t} f^{-1}(\Gamma_t)$ as follows

For any $p \in \Gamma_1$, $\xrightarrow{f} q = f(p)$, $\xleftarrow{f^{-1}}$ (some branch)

This "f^{-1}" is well-defined at least on $\mathcal{U}$ $\boxed{\text{Think}}$

Let $\mu_t(p)$ be the image of q_t under this (branch of) f^{-1}.

Then, $\mu_t(p_j) \neq \mu_t(p_k)$ for any $j \neq k$

[$\Rightarrow$ if so, same f^{-1} on $\mathcal{U}$ $\Rightarrow$ $p_j = p_k$]

Thus, $\Gamma_1 \xrightarrow{\mu_t} f^{-1}(\Gamma_t)$

$\{p_1, \cdots, p_d\} \longrightarrow d$-distinct point $\xrightarrow{f} q_t$ but $f^{-1}(q_t)$ has exactly d points

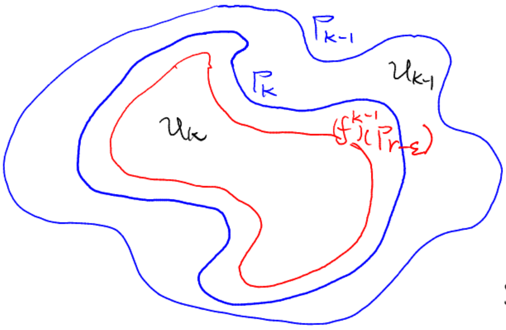
[$\Rightarrow$] If $\exists\, z \in Crit_f$ such that $f^n(z) \to \infty$ ($z \in A(\infty)$)

we want to show that J is disconnected.

Let k be the first integer so that $\mathcal{U}_k$ contains a critical point.

{ We may assume there is no critical point in $\mathcal{U}_0 \cup \{z \in \mathbb{C} \mid |z| \geq R\}$

For simplicity, only consider the case that $\mathcal{U}_k$ has only one critical point, q.



$\because$ The previous argument shows that

$\Gamma_{j\leq k} \cong S^1$ (analytically)

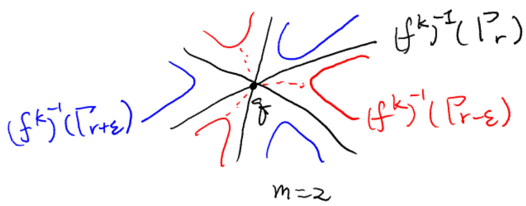
$\Rightarrow q \in (f^k)^{-1}(\Gamma_r)$ for some $r \in (0,1)$

Say, the order of f at q is $m \in \mathbb{N}$

i.e. $f(z) = f(q) + (z-q)^{m+1} h(z)$ $\to$ nowhere zero near q

$\Leftrightarrow f^{(\ell)}(q) = 0$ for $\ell \in \{1, \cdots, m\}$

Then, the local picture near q is



$w = (z-q)(h(z))^{\frac{1}{m+1}}$: conformal near q (change of coordinate analytically)

$f = f(q) + w^{m+1}$

Since $(f^k)^{-1}(\Gamma_{r-\varepsilon}) \cong S^1$, it follows that $(f^k)^{-1}(\Gamma_{r+\varepsilon})$ has at least $(m+1)$ components

$\Rightarrow$ W_{k+1} has at least $(m+1)$ components $\Rightarrow$ (by the structure of W_k)

$\Rightarrow$ J is disconnected. $\#$

The opposite case: all the critical points belong to $A(\infty)$

<u>defn</u> a compact set K in $\mathbb{C}$ is called totally disconnected if $\forall\, z_0 \in K$, and $\varepsilon > 0$. $\exists\, \underset{z_0}{\underset{\cup}{E}} \subset K$ such that $\underbrace{\text{diameter } E}_{\sup_{x,y\in E}|x-y|} < \varepsilon$ and $\operatorname{dist}(E, K\setminus E) > 0$

<u>rmk</u>
- $K = E \cup K\setminus E$ two disjoint compact set (using $\operatorname{dist}(\overline{E}, \overline{K\setminus E}) > 0$)
- given $z_0, z_1 \in K$. choose $\varepsilon = \frac{1}{2}|z_1 - z_0|$ $\Rightarrow$ $z_0 \in E$, $z_1 \in K\setminus E$

<u>Prop</u> $f(z)$: monic, $\deg = d \geq 2$. if $\operatorname{Crit}_f \subset A(\infty)$

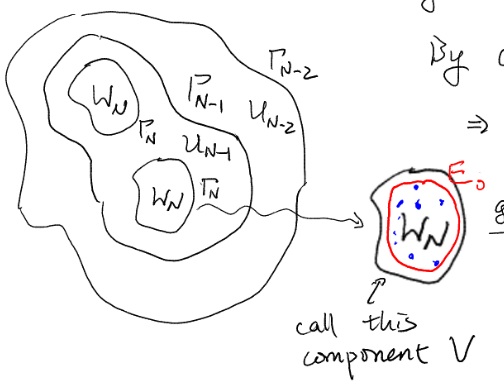
Then, $\mathcal{J}$ is totally disconnected. and $\mathcal{F} = A(\infty)$

<u>e.g.</u> $f(z) = z^2 + 1$

(try Wolfram Alpha julia set $f(z) = z^\wedge 2 + 1$, etc.)

Pf: Since Crit_f consists of finite number of points

Choose N large enough so that $W_N \cap \operatorname{Crit}_f = \phi$



By construction, each component of W_N is simply-connected

$\Rightarrow$ f^{-1} (also $(f^k)^{-1}$) is well-defined on each component of W_N (there are d choices of f^{-1})

$g_k = (f^k)^{-1}$

$\cdot\, z_0 \in \mathcal{J}$

Fix $z_0 \in \mathcal{J}$, $\exists$ some component of W_N, V. that contains infinitely many $\{f^k(z_0)\}_{k \geq N}$

For each such k, consider $g_k = (f^k)^{-1}$ that sends $f^k(z_0)$ to z_0

Since image $g_k \subset W_{N+k}$, $\{g_k\}$: normal family on V

$\Rightarrow$ $g_{k_j} \to g$ uniformly on compact subsets of V

For $w \in V \cap A(\infty)$. $g_{k_j}(w) \in W_{N+k_j} \cap A(\infty)$ $\Rightarrow$ $g(w) \in \mathcal{J}$

But the Julia set has no interior point $\Rightarrow$ $g(w) \equiv z_0$.

$\Rightarrow$ shrink V a little bit $\leadsto$ E_0: compact $\begin{cases} f^{k_j}(z_0) \in \text{interior of } E_0 \\ \partial E_0 \subset A(\infty) \end{cases}$

Consider $E_j = g_{k_j}(E_0)$: compact nbd of z_0

$\begin{cases} \text{Since } g_{k_j} \to z_0, \quad \text{diameter}(E_j) \to 0 \\ \text{Since } \partial E_0 \subset A(\infty), \quad \partial E_j \subset A(\infty) \Rightarrow \text{dist}(E_j, \mathcal{J} \setminus E_j) > 0 \end{cases}$

※

§4 the Mandelbrot set

apply the above criterions to the simplest case: $f_c(z) = z^2 + c$

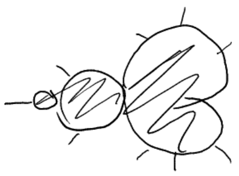
$\mathcal{J}_c$: Julia set of $f_c(z)$ $\qquad$ $\text{Crit}_{f_c} = \{0\}$ ← 1-point

connected if and only if $\{c, c^2+c, (c^2+c)^2+c, \cdots\}$ is bounded

totally disconnected $\quad$ if $f_c^n(0) \to \infty$

$c \in \mathbb{C}$ the "parameter" space

$\mathcal{M} = \{c \in \mathbb{C} \mid \{f_c^n(0)\}_{n\geq 1} \text{ is bounded} \Leftrightarrow \mathcal{J}_c \text{ is connected}\}$



thm i) $c \in \mathcal{M}$ iff $|f_c^n(0)| \leq 2 \quad \forall n \geq 1$

ii) $\mathcal{M}$ is a compact subset of $\{c \in \mathbb{C} \mid |c| \leq 2\}$

iii) $\mathbb{C} \setminus \mathcal{M}$ is connected.

pf: For i) & ii) $\qquad$ examine $|z| \geq c$

if $|c| > 2$ and $|z| = c$ $\qquad |f_c(z)| = |z^2 + c| \geq |c|^2 - |c| = (|c|-1)|z|$

$\Leftrightarrow \dfrac{|z|}{|f_c(z)|} \leq \dfrac{1}{|c|-1}$ for $|z| = c$

$f_c(z) \neq 0$ on $|z| \geq c$ $\Rightarrow$ maximum principle : $|f_c(z)| \geq (|c|-1)|z|$ $\quad \forall |z| \geq c$

It follows that $|f_c^n(0)| \geq (|c|-1)^{n-1}|c| \to \infty$

if $|c| \leq 2$.

let n: first integer so that $|f_c^n(0)| > 2$

$|f_c^{n+1}(0)| \geq |f_c^n(0)|^2 - |c| > |f_c^n(0)|^2 - |f_c^n(0)| = (|f_c^n(0)| - 1)|f_c^n(0)|$

$> |f_c^n(0)| > 2$

$|f_c^{n+2}(0)| \geq (|f_c^{n+1}(0)| - 1)|f_c^{n+1}(0)| > (|f_c^n(0)| - 1)^2 |f_c^n(0)|$

$\cdots \quad |f_c^{n+k}(0)| \geq (|f_c^n(0)| - 1)^k |f_c^n(0)|$

$\Rightarrow \mathcal{M} = \bigcap_{n\geq 1} \{c \in \mathbb{C} \mid |f_c^n(0)| \leq 2\}$

For iii) : consider $\mathbb{C} \setminus \mathcal{M} = \mathcal{U}_\infty \cup \mathcal{U}_1 \cup \cdots$ (components of $\mathbb{C} \setminus \mathcal{M}$)

$\mathcal{U}_1$: bounded and $\partial \mathcal{U}_1 \subset \mathcal{M}$ $\Rightarrow$ $|f_c^n(0)| \leq 2 \quad \forall c \in \partial \mathcal{U}_1$

$\Rightarrow$ maximum principle $\Rightarrow$ $\mathcal{U}_1 \subset \mathcal{M}$ $\to\leftarrow$

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