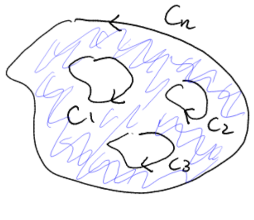
XIV. conformal mapping between multiply-connected region
application of normal family

§1. multiply-connected region

$\Omega = \hat{\mathbb{C}} \setminus \{E_1 \cup \cdots \cup E_n\}$ open & connected

E_j : closed & connected, $E_j \neq$ point, $\infty \in E_n$, $\hat{\mathbb{C}} \setminus E_j$: simply-connected



After applying Riemann mapping a couple times, we may assume that the Dirichlet problem can be solved on Ω,

Moreover, we may assume that each boundary curve is the image of $|z|=1$ under an analytic map.

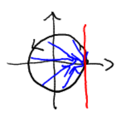
<u>lemma</u> (argument principle) $f(z)$: non-constant analytic function on $\bar{\Omega}$

then, $\displaystyle \frac{1}{2\pi i} \int_{\cup C_j} \frac{f'(z)}{f(z)-w_0}\, dz = \#\{z \in \Omega \mid f(z) = w_0\} + \frac{1}{2} \#\{z \in C_j \mid f(z) = w_0\}$

with multiplicity counted

(namely, we allow $f(C_j)$ passing through w_0)

pf: key case $f(z) = z-1$ on the unit disk



$$\frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z-1} = \frac{1}{2\pi} \int_{|z|=1} d\arg(z-1) = \frac{1}{2}$$

$\#$

$\boxed{\text{goal}}$ map C_1 to $|z| = e^{\lambda}$.

C_n to $|z| = 1$

C_j to some arc

By Perron method. let $w_j(z)$ be the harmonic function with $w_j|_{C_j} = 1$, $w_j|_{C_{i \neq j}} = 0$

By the maximum principle / uniqueness, $\displaystyle\sum_{j=1}^{n} w_j \equiv 1$

$F = e^{u+iv}$ $\quad u = \displaystyle\sum_{j=1}^{n-1} \lambda_j w_j$ $\quad \lambda_1, \cdots, \lambda_{n-1}$: to be determined

Compatibility condition for constructing v :

$$\sum_{j=1}^{n-1} \lambda_j \int_{C_1} \left(-\frac{\partial w_j}{\partial y} dx + \frac{\partial w_j}{\partial x} dy\right) = 2\pi$$

$$\sum_{j=1}^{n-1} \lambda_j \int_{C_k} \left(-\frac{\partial w_j}{\partial y} dx + \frac{\partial w_j}{\partial x} dy\right) = 0 \qquad \text{for } 1 < k < n$$

$$\left(\Rightarrow \sum_{j=1}^{n-1} \lambda_j \int_{C_n} \left(-\frac{\partial w_j}{\partial y} dx + \frac{\partial w_j}{\partial x} dy\right) = -2\pi \quad \text{by Green's theorem}\right)$$

Solving

$$\begin{bmatrix} \int_{C_1} \frac{\partial w_1}{\partial n} ds & \cdots & \int_{C_1} \frac{\partial w_{n-1}}{\partial n} ds \\ \vdots & \ddots & \vdots \\ \int_{C_{n-1}} \frac{\partial w_1}{\partial n} ds & \cdots & \int_{C_{n-1}} \frac{\partial w_{n-1}}{\partial n} ds \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} = \begin{bmatrix} 2\pi \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

claim the $(n-1)\times(n-1)$-matrix is invertible

$$\left(\begin{array}{l}\text{If NOT, } \exists\,(\lambda_1,\cdots,\lambda_{n-1}) \text{ not all zero , such that} \\ u=\sum_{j=1}^{n-1}\lambda_j w_j \text{ has a well-defined harmonic conjugate on } \Omega \\ \Omega \xrightarrow{u+iv} \end{array}\right)$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\cup C_j}\frac{f'(z)}{f(z)-w_0}\,dz = 0 \quad \rightarrow\leftarrow$$

$\rightsquigarrow$ $\exists!\,(\lambda_1,\cdots,\lambda_{n-1})$ solving the above linear system

let $u=\sum_{j=1}^{n-1}\lambda_j w_j$ and v = harmonic conjugate

not single valued , well-defined up to $2\pi\mathbb{Z}$

Consider $F=e^{u+iv}:\overline{\Omega}\to\mathbb{C}$

By construction, $F(C_j)\subset\{|w|=e^{\lambda_j}\}$ $\quad F(C_n)\subset\{|w|=1\}$

$$\frac{1}{2\pi i}\int_{C_1}\frac{F'(z)}{F(z)}\,dz = 1\,,\quad \frac{1}{2\pi i}\int_{C_n}\frac{F'(z)}{F(z)}\,dz = -1\,,\quad \frac{1}{2\pi i}\int_{C_j}\frac{F'(z)}{F(z)}\,dz = 0$$

for $1<j<n$

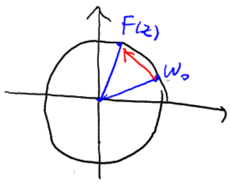
$$\Rightarrow \frac{1}{2\pi i}\int_{C_1}\frac{F'(z)}{F(z)-w_0}\,dz = \begin{cases}1 & \text{for } |w_0|<e^{\lambda_1}\\ 0 & \text{for } |w_0|>e^{\lambda_1}\end{cases},\quad \frac{1}{2\pi i}\int_{C_n}\frac{F'(z)}{F(z)-w_0}\,dz = \begin{cases}-1 & |w_0|<1\\ 0 & |w_0|>1\end{cases}$$

$$\frac{1}{2\pi i}\int_{C_j}\frac{F'(z)}{F(z)-w_0}\,dz = 0 \quad \text{for } |w_0|\neq e^{\lambda_j} \quad \text{for } 1<j<n$$

Hence, $1<e^{\lambda_1} \iff 0<\lambda_1$, also $0\le\lambda_j\le\lambda_1$ for $1<j<n$

For any $|w_0|=e^{\lambda_1}$, the construction of u & v implies that z_0 is mapped at least once for $F|_{C_1}$

claim $\displaystyle\sum_{j=1}^{n}\frac{1}{2\pi i}\int_{C_j}\frac{F'(z)}{F(z)-w_0}\,dz = \frac{1}{2}$



$\arg(F(z)-w_0)$
$=\frac{1}{2}(\arg(F(z))+\arg w_0)$

$$\frac{1}{2\pi i}\int_{C_1}\frac{F'(z)}{F(z)-w_0}\,dz = \frac{1}{2\pi}\int_{C_1}d\arg(F(z)-w_0)$$
$$= \frac{1}{4\pi}\int_{C_1}d\arg F(z) = \frac{1}{2}$$

$$\frac{1}{2\pi i}\int_{C_j}\frac{F'(z)}{F(z)-w_0}\,dz = 0 \quad \left[\begin{array}{ll}\text{if } |w_0|\neq e^{\lambda_j}, & \text{direct.}\\ \text{if } |w_0|=e^{\lambda_j}, & \text{use } d\arg(F(z)-w_0)=\frac{1}{2}d\arg F(z)\end{array}\right]$$

$\Rightarrow$ $F^{-1}(w_0)$ has only 1 preimage , in C_1

Same story holds for $|w|=1 \longleftrightarrow C_n$

The image of C_k under F (when $2<k<n$)

Above discussion shows that $0<\lambda_k<\lambda_1$

For any $|w_0|=e^{\lambda_k}$ $\quad \displaystyle\sum_{j=1}^{n}\int_{C_j}\frac{F'(z)}{F(z)-w_0}\,dz = 1+\cdots+0+0 = 1$

Three possibilities $\begin{cases} 2 \text{ points in the boundary curves} \\ 1 \text{ points in the interior} \\ 1 \text{ point in the boundary curves} \\ \quad \text{with multiplicity two} \end{cases} \longmapsto w_0$

Remember that $\log F(z)$ (also $\arg F(z)$) is well-defined on C_k

$C_k \xrightarrow{F}$ has index 0 around the origin

$\Rightarrow$ The image of C_k under $u + iv$ can has only two critical points, one max and one min, and the length is less than 2π.

constant

Uniqueness. $u = \mathrm{Re} \log F \longleftrightarrow F$ up to a rotation

initial condition to construct v

$\Omega \longleftrightarrow e^{\lambda_1}, \dots e^{\lambda_{n-1}}$

position & length of C_k, $1 < k < n$

$n-1 + 2(n-2) - 1 = 3n-6$ dimension of freedom

§2 application of normal family [G, §2 of ch. XII]

<u>recall</u> (Marty) $\mathcal{F}$: family of meromorphic function on Ω
$\mathcal{F}$ is normal if and only if $\rho(f) = \dfrac{2|f'|}{1+|f|^2}$ is locally bounded

<u>Q</u> How to handle "non-normal" family ?
$\{f_n\}$: sequence of meromorphic function, NOT NORMAL
$\Rightarrow \exists \{w_n\} \subset K$: compact $\subset \Omega$ such that $\rho(f_n)(w_n) \xrightarrow{n\to\infty} \infty$

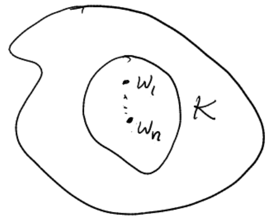
<u>strategy</u> (blow-up) construct a rescaled limit which captures the behavior of f_n near w_n

<u>naive idea</u>

Let $M_n = \rho(f_n)(w_n)$

Consider $g_n(w) = f_n(w_n + \frac{w}{M_n})$

(shift $w_n \mapsto 0$, zoom in with ratio M_n)



$\Rightarrow g_n'(w) = f_n'(w_n + \frac{w}{M_n})\frac{1}{M_n} \Rightarrow \rho(g_n(0)) = \frac{1}{M_n}\rho(f_n(w_n)) = 1$

$\Rightarrow \rho(g_n(w)) = \frac{1}{M_n}\rho(f_n(w_n + \frac{w}{M_n}))$ ⇜ have to control it near $w=0$

<u>modified argument</u> May assume $\overline{D} = \{z \in \mathbb{C} \mid |z| \le 1\} \subset \Omega$ (By rescaling)
and $w_n \xrightarrow{n\to\infty} 0$

Need a way to relate the value of $\rho(f_n)$ at different points

Consider $\rho(f_n)(z)\cdot(1-|z|)$ assume its maximum achieves at z_n

Since $\rho(f_n)(w_n) \to \infty$ and $w_n \to 0$, $\max_{\overline{D}} \rho(f_n)(z)(1-|z|) \to \infty$

$$\rho(f_n)(z_n) \ge \rho(f_n)(z_n)\cdot(1-|z_n|) \Rightarrow \rho(f_n)(z_n) \to \infty$$

Denote $\rho(f_n)(z_n)$ by M_n

Again, consider $g_n(w) = f_n(z_n + \frac{w}{M_n}) \Rightarrow \rho(g_n)(0) = \dfrac{\rho(f_n)(z_n)}{M_n} = 1$

$$\rho(g_n(w)) = \frac{\rho(f_n)(z_n + \frac{w}{M_n})}{M_n} \le \frac{1}{M_n}\frac{(1-|z_n|)M_n}{1-|z_n + \frac{w}{M_n}|}$$

$$\le \frac{1}{1 - \frac{w}{(1-|z_n|)M_n}} \quad \text{for } |w| < M_n(1-|z_n|)$$

(on any $|w| < R \Rightarrow \rho(g_n(w))$ is uniformly bounded)

By Marty, $g_n(w) \to g(w)$ on $\mathbb{C}$
(uniformly on any compact subset, wrt $\hat{\mathbb{C}}$-metric)

Also, $\rho(g)(0) = 1$, $\rho(g)(w) \le 1$

thm (Zalcman's lemma) $\{f_n\}$: meromorphic on Ω, not normal

Then, $\exists \{z_n\} \to z_0 \in \Omega$, $M_n > 0 \to \infty$

such that $g_n(w) = f_n(z_n + \frac{w}{M_n}) \to g(w)$: non-constant meromorphic function on $\mathbb{C}$. with $\rho(g)(0) = 1$

$\rho(g)(w) \leq 1$

thm (Montel) $\mathcal{F}$: family of meromorphic function on Ω.

Suppose that $\mathcal{F}$ omits three values in $\hat{\mathbb{C}}$, then it is normal

pf: By composing with an element of $SL(2;\mathbb{C})$, we may assume

$\mathcal{F}$ omits $\{0, 1, \infty\}$,

Hence, $\mathcal{F}$ consits of nowhere zero analytic functions.

If $\mathcal{F}$ is not normal, $\exists \bar{D} \subset \Omega$, $\{f_n\} \in \mathcal{F}$, $w_n \in D \to 0$

such that $\rho(f_n)(w_n) \to \infty$

Denote $\mathcal{J} = \{f_1, \cdots, f_n, \cdots\}$

By Zalcman, $\exists g$: rescaled limit of $\mathcal{J}$

Similarly $\mathcal{J}_k = \{f_1^{1/2^k}, \cdots, f_n^{1/2^k}, \cdots\}$ is not normal

$\Rightarrow \exists g_k$: rescaled limit of $\mathcal{J}_k$. (meromorphic on $\mathbb{C}$)

properties of g_k: $\rho(g_k)(0) = 1$ $\rho(g_k)(w) \leq 1$

g_k must be analytic

g_k omit all the 2^k-th roots of unity

(argument principle)

But $\mathcal{G} = \{g, g_1, \cdots, g_k, \cdots\}$ is normal

$\therefore \exists$ limit h : $\rho(h)(0) = 1$, h : analytic

(meromorphic on $\mathbb{C}$) h omits all the 2^k-th roots of unity $\forall k$

By open mapping, $h(\mathbb{C}) \subset \{|z| < 1\}$

or $\{|z| > 1\}$

If $h(\mathbb{C}) \subset \{|z| < 1\}$ $\Rightarrow$ Liouville h = constant

$\Rightarrow \rho(h)(0) = 0 \rightarrow\leftarrow$

If $h(\mathbb{C}) \subset \{|z| > 1\}$ $\Rightarrow$ apply Liouville to $1/h$

$\Rightarrow \rho(h)(0) = \rho(1/h)(0) = 0 \rightarrow\leftarrow$

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(almost finish the proof for Great Picard & Little Picard theorem)