XIII. Dirichlet problem, Perron's method, conformal equivalence for annulus-type region

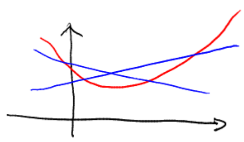
goal: study conformal equivalence for non-simply-connected domain by studying the boundary value problem for harmonic maps.

§1 subharmonic functions [Ahlfors, §4.1 of ch.6]

In dimension one, $\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = ax+b$: straight lines

$\frac{\partial^2 u}{\partial x^2} > 0$: convex

characterized by using straight lines



② Perron's method

see the picture

defn $v \in \mathcal{C}(\Omega;\mathbb{R})$ is said to be subharmonic if $\forall\, \Omega' \subset \Omega$, u: harmonic on Ω', $v-u$ satisfies the maximum principle.

(Namely, if $v-u$ achieves maximum in Ω', it must be a constant)

lemma if $v \in \mathcal{C}^2(\Omega)$, $\Delta v \geq 0$, then v is subharmonic $\boxed{\text{HW}}$

thm $v \in \mathcal{C}(\Omega)$ is subharmonic if and only if

$$v(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\, d\theta \quad \text{for any } \overline{B(z_0;r)} \subset \Omega$$

pf: $\Leftarrow$ $\underset{\text{achieves maximum on } \Omega'}{(v-u)(z_0)} \leq \frac{1}{2\pi}\int_0^{2\pi}(v-u)(z_0+re^{i\theta})\, d\theta \cdots \Rightarrow (v-u)^{-1}(z_0)$ is open and closed in Ω'

$\Rightarrow$ Consider $u(z_0+w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-|w|^2}{|re^{i\theta}-w|^2}\, v(z_0+re^{i\theta})\, d\theta \quad \forall\, w \in B(0;r)$

u: harmonic on $B(z_0;r)$, continuous on $\overline{B(z_0;r)}$

$u(z) = v(z)$ on $\partial B(z_0;r)$

Hence $(v-u)(z) \leq \max\limits_{\partial B(z_0;r)} (v-u) = 0$

$$v(z_0) \leq u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} v(z_0+re^{i\theta})\, d\theta$$

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prop i) kv $(k \geq 0)$, v_1+v_2 are also subharmonic

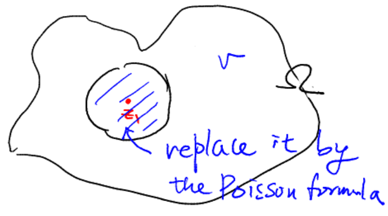
ii) $\max(v_1,v_2)$ is subharmonic

iii) Given any $\overline{B(z_1;r)} \subset \Omega$, define $\tilde{v}$ by $\begin{cases} v & z \in \Omega \setminus B(z_1;r) \\ \text{Poisson integral} & z \in B(z_1;r) \end{cases}$

$\tilde{v}$ is also subharmonic

pf: i) by the thm above

ii) Let $v(z) = \max\{v_1(z), v_2(z)\}$



Given u: harmonic on Ω', assume $v-u$ achieves maximum at $z_0 \in \Omega'$

Say $v(z_0) = v_1(z_0)$

$$\forall z \in \Omega' \quad \underset{\underset{v_1(z)-u(z)}{\vee\!\!|}}{v(z)-u(z)} \le v(z_0)-u(z_0) = v_1(z_0)-u(z_0)$$

Thus, all the equality holds, and $v_1 - u$ is a constant on Ω'

Then so is $v-u$.

iii) Note that $v \le \tilde{v}$ $\forall z \in \Omega$

Given u: harmonic on Ω', assume $\tilde{v}-u$ achieves maximum at $z_0 \in \Omega'$

$$\tilde{v}(z) - u(z) \le \tilde{v}(z_0) - u(z_0)$$

case 1 if $z_0 \notin B(z_1;r)$ $\Rightarrow$ $\tilde{v}(z_0) = v(z_0)$

$\Rightarrow$ $v(z)-u(z) \le \tilde{v}(z) - u(z) \le \tilde{v}(z_0) - u(z_0) = v(z_0)-u(z_0)$ $\forall z \in \Omega$

the subharmonic property of v implies that all the inequalities are equality.

case 2 if $z_0 \in B(z_1;r)$, similarly, $\tilde{v}(z)-u(z) \equiv \tilde{v}(z_0) - u(z_0)$ on $\Omega' \cap B(z_1;r)$

equality to $\partial B(z_1;r) \cap \Omega'$

back to case 1

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§2 solving the Dirichlet problem [Ahlfors, § 4.2 of ch. 6]

$\zeta \in \partial\Omega$ Ω: bounded and connected

f: function on $\partial\Omega$ (not necessarily being continuous)

Asume boundedness, $|f(\zeta)| \le M$

defn $\mathcal{B}(f) = \{ v$: subharmonic on Ω and $\limsup_{z \to \zeta} v(z) \le f(\zeta)$ $\forall \zeta \in \partial\Omega \}$

Perron's strategy for solving the Dirichlet problem

1° consider $u(z) = \sup_{v \in \mathcal{B}(f)} v(z)$, $u(z)$: harmonic?

2° $\lim_{z \to \zeta} u(z) = f(\zeta)$ under suitable condition?

[For 1°] Lemma $u(z)$ is harmonic

Pf: step 0) By the maximum principle, $v(z) \le M$ $\forall v \in \mathcal{B}(f)$

$\Rightarrow u(z) \le M$

(Detail: For $v \in \mathcal{B}(f)$ and $\varepsilon > 0$. consider $E = \{ z \in \Omega \mid v(z) \ge M + \varepsilon \}$

E: closed in $\overline{\Omega}$ $\Rightarrow$ E is compact

$\Rightarrow$ v achieves maximum on E $\rightarrow\leftarrow$ Namely $E = \phi$)

step 1) $\forall z_0 \in \Omega$, u: harmonic on a neighborhood of z_0?

Choose $\Delta = \overline{B(z_0;r)} \subset \Omega$

$\exists \{v_n\} \subset \mathcal{B}(f)$ such that $\lim_{n\to\infty} v_n(z_0) = u(z_0)$

(Idea: subharmonic ⇝ harmonic ⇝ applying Harnack)

Take $V_n = \max\{v_1, \dots, v_n\}$ then $\{V_j\}$: non-decreasing sequence of subharmonic functions.

Still, $\{V_j\} \in \mathcal{B}(f)$ and $V_j(z_0) \to u(z_0)$

Let $\tilde{V}_j = \begin{cases} V_j & \text{on } \Omega \setminus \Delta \\ \text{by Poisson formula} & \text{on } \Delta \end{cases} \in \mathcal{B}(f)$

Since $V_j \leq \tilde{V}_j$, we still have $\tilde{V}_j(z_0) \to u(z_0)$

Non-decreasing of $\tilde{V}_j$ on Δ ?

$\forall z \in \Delta \qquad \tilde{V}_j(z) - \tilde{V}_{j+1}(z) \leq \max\limits_{\partial\Delta} \tilde{V}_j - \tilde{V}_{j+1} = \max\limits_{\partial\Delta} V_j - V_{j+1} \leq 0$

By Harnack, $\tilde{V}_j \to \mathcal{U}$: harmonic on Δ and uniformly on compact subset of Δ

clearly, $\mathcal{U}(z_0) = u(z_0)$

step 2) $\mathcal{U} = u$ on Δ ?

Given any $z_1 \in \Delta$, choose $w_j \in \mathcal{B}(f)$ with $w_j(z_1) \to u(z_1)$

subharmonics achieve $u(z_0)$ & $u(z_1)$ $\rightsquigarrow$ Let $\overline{w}_j = \max\{w_j, v_j\}$ $\quad$ (so $\overline{w}_j(z_1) \to u(z_1)$ and $\overline{w}_j(z_0) \to u(z_0)$)

Similarly, take $W_n = \max\{\overline{w}_1, \dots, \overline{w}_n\}$

and take $\tilde{W}_n = \begin{cases} W_n & \text{on } \Omega \setminus \Delta \\ \text{by Poisson} & \text{on } \Delta \end{cases}$

Again by Harnack, $\tilde{W}_n \to \mathcal{U}_1$: harmonic on Δ

and $\mathcal{U}_1(z_1) = u(z_1)$

Since $\tilde{V}_n \leq \tilde{W}_n$, $\mathcal{U} \leq \mathcal{U}_1 \leq u$ on Δ

$\Rightarrow \mathcal{U} - \mathcal{U}_1 \leq 0$ on Δ but $= 0$ at z_1

$\Rightarrow \mathcal{U} \equiv \mathcal{U}_1$. $\Rightarrow \mathcal{U}(z_1) = \mathcal{U}_1(z_1) = u(z_1)$ $\Rightarrow \mathcal{U} \equiv u$ on Δ #

[For 2°] Dirichlet problem does not always admit a solution

e.g. $\Omega = D \setminus \{0\}$, $f = 0$ on ∂D & $f(0) = 1$ $\boxed{\text{HW}}$

lemma (barrier function) $\zeta_0 \in \partial\Omega$

If $\exists\, w$: continuous on $\overline{\Omega}$, harmonic on Ω,

$w(\zeta_0) = 0$, and positive elsewhere on $\partial\Omega$

$\Rightarrow \lim\limits_{z \to \zeta_0} u(z) = f(\zeta_0)$

pf: 

goal $\limsup\limits_{z \to \zeta_0} u(z) \leq f(\zeta_0) + \varepsilon \quad \forall \varepsilon > 0$ $\quad$ by constructing a harmonic function whose boundary value is greater than the RHS.

$\exists\, V$: neighborhoof of ζ_0 in $\mathbb{C}$

such that $|f(\zeta) - f(\zeta_0)| < \varepsilon \quad \forall \zeta \subset V \cap \partial\Omega$

let $\underbrace{w_0 = \min\limits_{\partial\Omega \setminus V} w(z) > 0}$

Consider $W(z) = f(\zeta_0) + \dfrac{M - f(\zeta_0)}{w_0}\, w(z) + \varepsilon$: harmonic

When $\zeta \in \partial\Omega \cap V$, $\quad W(\zeta) > f(\zeta_0) + (\text{non-negative}) + \varepsilon \geq f(\zeta)$

When $\zeta \notin \partial\Omega \setminus V$, $\quad W(\zeta) > f(\zeta_0) + M - f(\zeta_0) + \varepsilon$

Hence, the maximum principle implies that

$$v(z) \le W(z) \qquad \forall v \in \mathcal{B}(f)$$

$$\Rightarrow \limsup_{z\to\zeta_0} u(z) \le \limsup_{z\to\zeta_0} W(z) = W(\zeta_0) = f(\zeta_0) + \varepsilon$$

For $\liminf\limits_{z\to\zeta_0} u(z) \ge f(\zeta_0) - \varepsilon$

need $S\,w(\zeta) \ge f(\zeta_0) - \varepsilon - f(\zeta)$

but $-f(\zeta) \le M$

$\Rightarrow S\,w(\zeta) \ge M + f(\zeta_0) - \varepsilon$

Consider $\quad V(z) = f(\zeta_0) - \varepsilon - \overbrace{\frac{M + f(\zeta_0)}{w_0}}^{S} w(z)$

$\zeta \in \partial\Omega \cap V$, $\quad V(\zeta) \le f(\zeta_0) - \varepsilon < f(\zeta)$

$\zeta \notin \partial\Omega \cap V$, $\quad V(\zeta) \le f(\zeta_0) - \varepsilon - \underbrace{M - f(\zeta_0)}_{S\,w(\zeta)} = -M - \varepsilon < f(\zeta)$

$\Rightarrow V(z) \in \mathcal{B}(f)$

$$\liminf_{z\to\zeta_0} u(z) \ge \liminf_{z\to\zeta_0} V(z) = f(\zeta_0) - \varepsilon$$

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<u>thm</u> if $\forall \zeta_0 \in \partial\Omega$ $\exists$ interval $\subset \mathbb{C}\setminus\overline{\Omega}$ such that $\zeta = \partial(\text{interval})$

then the Dirichlet problem can be solved for Ω.

about the shape of $\partial\Omega$

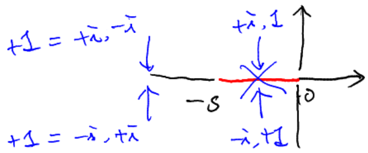
pf:

By rotation and translation, may assume $\zeta_0 = 0$, $\zeta_1 = -s < 0$

$-s$ —— 0 : NOT in Ω. $\quad 0 \in \partial\Omega$

Consider $\quad \sqrt{\frac{z}{z+s}} = \sqrt{z}\,\frac{1}{\sqrt{z+s}}$ analytic on $\mathbb{C}\setminus\{\text{negative real}\}$

$\Rightarrow$ analytic on $\mathbb{C}\setminus\{[-s,0]\}$



$$\frac{z}{z+s} = -t \Rightarrow z = -tz - st \Rightarrow z = -s\frac{t}{1+t} \in [-s, 0]$$

$\Rightarrow \operatorname{Re}\sqrt{\frac{z}{z+s}}$ is a barrier function

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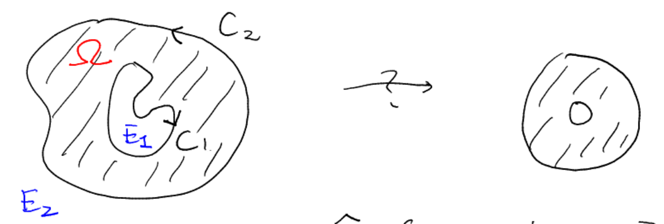
§3 conformal mapping of annulus-type region [Ahlfors, §5.1 of ch. 6]

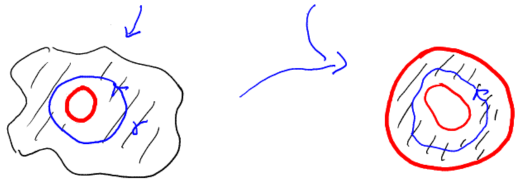


annulus

$\Omega = \hat{\mathbb{C}} \setminus \{E_1 \cup E_2\}$ $\quad E_1, E_2$: closed, connected

open & connected

such that $\hat{\mathbb{C}} \setminus E_1$, $\mathbb{C} \setminus E_2$: simply connected (hence conformal equivalent to D)



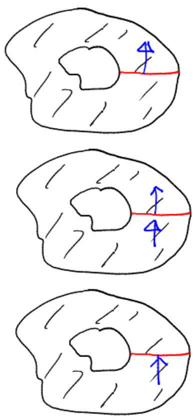
In any case, we may assume C_1 & C_2 are embedded smooth loop in $\mathbb{C}$.

$F : \Omega \to$ annulus ?

$F = u + iv$: hard to describe the boundary condition

attempt $\quad F = e^f = e^{u+iv}$

> u: constant on boundary
> But v: not single-valued
> travel along γ : the value of v differs by 2π



$u + iv$:(locally) analytic

[rmk well-defined on the "universal covering" of Ω
a "helix-type" domain]

conditions $\quad u_x = v_y$, $u_y = -v_x$ $\qquad u|_{C_1} = \lambda \qquad u|_{C_2} = 0$

u can be found by solving the Dirichlet problem

Also, by reflection principle, we may extend u harmonically across C_1 & C_2 $\boxed{\text{HW: BONUS}}$

Construct v from u ?

$u \rightsquigarrow$ associate the differential $u_x\,dx + u_y\,dy$

$\leftsquigarrow$ recover by line integral, up to a constant

$$v_x\,dx + v_y\,dy = -u_y\,dx + u_x\,dy$$

Discussion for $-u_y\,dx + u_x\,dy$ =

$$\int_{C_2} -u_y\,dx + u_x\,dy - \int_{C_1} -u_y\,dx + u_x\,dy \overset{\text{Green's}}{\underset{\text{thm}}{=}} \iint_{\Omega} (u_{xx} + u_{yy})\,dx\,dy = 0$$

$$(P, Q) \xrightarrow{\text{2D curl}} Q_x - P_y$$

Similarly. $\quad \displaystyle\int_{\gamma} -u_y\,dx + u_x\,dy = \int_{C_1} -u_y\,dx + u_x\,dy$

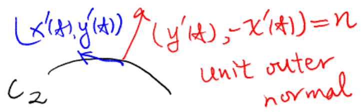
Observe (by topology) that γ (equivalently, C_1 & C_2) is the only obstruction for the closed loop property of $-u_y\,dx + u_x\,dy$.

Hence, if $\boxed{\int_{\gamma \text{ (also } C_2)} -u_y\,dx + u_x\,dy = -2\pi}$

then $\int -u_y\,dx + u_x\,dy$ defines a multi-valued harmonic function on $\overline{\Omega}$, with ambiguity $2\pi\mathbb{Z}$.

$C_2 = (x(t), y(t))$ $\quad t$: arc-length

$$\int_{C_2} -u_y\,dx + u_x\,dy = \int_0^{\ell} \left(-\frac{\partial u}{\partial y} x'(t) + \frac{\partial u}{\partial x} y'(t)\right) dt$$

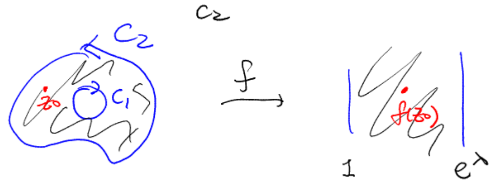


$$= \int_0^{\ell} \left(\text{partial derivative of } u \text{ along } (y'(t), -x'(t))\right) dt$$

$$= \int_{C_2} \frac{\partial u}{\partial n}\, ds$$

We have to show that $\int_{C_2} \frac{\partial u}{\partial n}\, ds \neq 0 \rightarrow$ rescaling (on λ) to 2π

If $\int_{C_2} \frac{\partial u}{\partial n}\, ds = 0 \Rightarrow f = u + iv$ is a well-defined analytic function on Ω



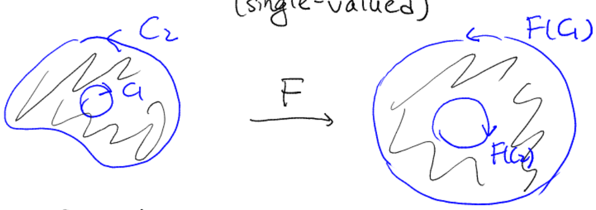
$$\int_{f(C_1)} \frac{dw}{w - f(z_0)} = 0 = \int_{f(C_2)} \frac{dw}{w - f(z_0)}$$

$$\Leftrightarrow \int_{C_1 \cup C_2} \frac{f'(z)\,dz}{f(z) - f(z_0)} = 0 \quad \to \leftarrow$$

Thus, we have a well-defined analytic function (single-valued), $F = e^{u+iv} : \overline{\Omega} \to \mathbb{C}$

$C_1 \mapsto |z| = e^{\lambda}$

$C_2 \mapsto |z| = 1$



conformal equivalence ?

$$\int_{C_j} \frac{F'(z)}{F(z) - w_0}\, dz = ?$$

Start with $\frac{1}{2\pi i} \int_{C_2} \frac{F'(z)}{F(z)}\, dz = \frac{1}{2\pi i} \int_{C_2} (u' + i v')\, dz$ $\quad \to 0 \quad -2\pi$

$= -1$

Similarly $\frac{1}{2\pi i} \int_{C_1} \frac{F'(z)}{F(z)}\, dz = +1$

Hence, if $|w_0| < 1$, $\frac{1}{2\pi i} \int_{C_1 \cup C_2} \frac{F'(z)}{F(z) - w_0}\, dz = 0 \Rightarrow w_0 \notin F(\Omega)$

Similar argument shows that if $|w_0| > e^{\lambda} \Rightarrow w_0 \notin F(\Omega)$

argument principle

if $1 < |w_0| < e^{\lambda} \Rightarrow$ exactly one root

(also imply that $1 < \lambda$)