XII. conformal mapping of polygons, harmonic function

§1 Schwarz - Christoffel formula [Ahlfors, §2.2 of ch. 6]

i) P : polygon $\quad D$: unit disk

<u>lemma</u> a conformal equivalence $f : P \to D$ extends to a homeomorphis $F : \bar{P} \to \bar{D}$

pf: According to the discussion last time, it extends to a continuous map $F : \bar{P} \to \bar{D}$ (note that the discussion works at the vertex as well)

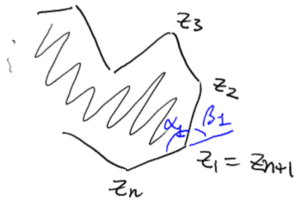
Similarly, $g = f^{-1} : D \to P$ admits a continuous extension $G : \bar{D} \to \bar{P}$

For any $z \in P$, $G \circ F(z) = z$. For any $w \in D$, $F \circ G(w) = w$

If $z \in \partial P$. $\exists z_n \in P$ such that $z_n \to z$ $\Rightarrow$ $G \circ F(z) = z$

By the same token, $F \circ G(w) = w$ $\forall w \in \partial D$ ※

P :



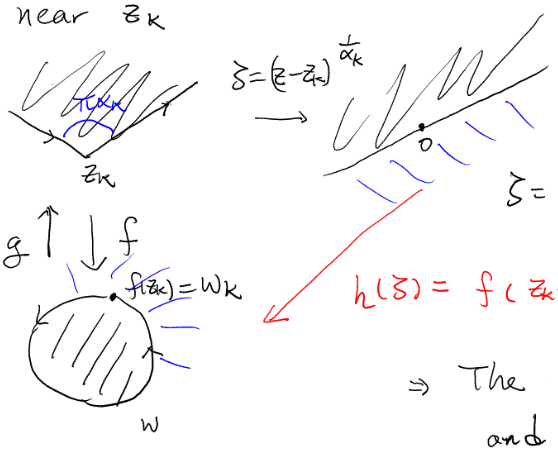
$z_1, z_2, \cdots, z_n, z_{n+1} = z_1$: vertices of P in counter clockwise order ($z_j \to z_{j+1}$: P is on the left hand side)

$\pi \alpha_j$ = angle at z_j = $\arg \dfrac{z_{j+1} - z_j}{z_j - z_{j-1}}$ $\in (0, 2\pi)$

$\pi \beta_j$ = outer angles = $\pi(1 - \alpha_j)$ $\qquad$ note that $\sum_{j=1}^{n} \beta_j = 2$

$\beta_j \in (-1, 1)$

ii) locally, near z_k



Consider $(z - z_k)^{\frac{1}{\alpha_k}}$ (some branch)

The map "flattens" the corner.

$\zeta = (z - z_k)^{\frac{1}{\alpha_k}} \in$ near 0, and some half-plane

$\Rightarrow$ $z = z_k + \zeta^{\alpha_k}$ (with suitable branch)

$\Rightarrow$ The Schwarz reflection principle applies, and $h(\zeta)$ is analytic on a neighborhood of 0

Moreover, since h is originally injective on the half plane $\Rightarrow$ $h'(0) \neq 0$

Hence, we can invert h for w near w_k and ζ near 0

$\Rightarrow$ $\zeta = (w - w_k) A_k(w)$ $\qquad$ $A_k(w)$: analytic near w_k) and $A_k(w_k) \neq 0$

It follows that $w \longmapsto \zeta = (w - w_k) A_k(w)$
$= (z - z_k)^{\frac{1}{\alpha_k}}$

$\Rightarrow$ $g(w)$ near w_k is $z_k + (w - w_k)^{\alpha_k} \underbrace{(A_k(w))^{\alpha_k}}_{\text{nonzero, analytic near } w_k}$

iii) global (putting together the local picture at each k) ?

$g'(w)$ is nonzero for $w \in D$, also for $w \in \partial D \setminus \{w_1, \dots, w_n\}$

near z_k, $g(w) = z_k + (w-w_k)^{\alpha_k}(A_k(w))^{\alpha_k}$

$$\Rightarrow g'(w) = \alpha_k (w-w_k)^{\alpha_k - 1}(A_k(w))^{\alpha_k} + (w-w_k)^{\alpha_k} \frac{d}{dw}(A_k(w))^{\alpha_k}$$

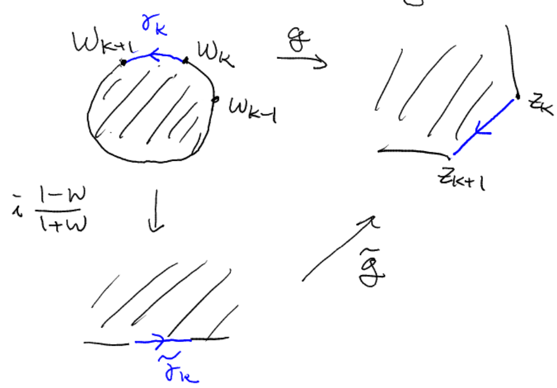
$$\Rightarrow (w-w_k)^{\beta_k} g'(w) = \alpha_k (A_k(w))^{\alpha_k} + (w-w_k)\frac{d}{dw}(A_k(w))^{\alpha_k}$$

It follows that $B(w) = g'(w) \prod_{k=1}^{n} (w-w_k)^{\beta_k}$ is analytic, and nowhere zero on $\overline{D}$

[More precisely, $(w-w_k)^{\beta_k} g'(w)$ has a removable singularity at $w=w_k$, and has nonzero value at $w=w_k$]

iv) claim $B(w) =$ constant

Consider $\log B(w) = \log|B(w)| + i \arg(B(w))$



$w_k = e^{i\theta_k}$ $\quad w_{k+1} = e^{i\theta_{k+1}}$

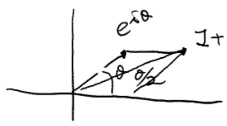
Consider $w = e^{i\theta}$ $\quad \theta_j < \theta < \theta_{j+1}$ (for some j)

$\tilde{g}'|_{\tilde{\gamma}_k}$ has constant argument / angle

$\arg \tilde{g}'|_{\tilde{\gamma}_k} = c_k' = \arg g|_{e^{i\theta}} - (c_k - \theta)$

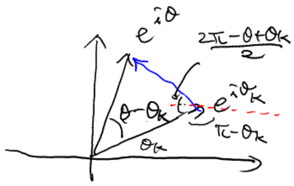
independent of $\theta \in (\theta_k, \theta_{k+1})$

$\left[\frac{d}{dw}\Big|_{e^{i\theta}} \left(i\frac{1-w}{1+w}\right) = \frac{-2i}{(1+e^{i\theta})^2} \overset{\arg}{\Rightarrow} c_k - \arg(1+e^{i\theta})^2 = c_k - \theta \right]$



$\Rightarrow \arg g'(w)|_{e^{i\theta}} = c_k + c_k' - \theta$

$\arg \prod_k (w-w_k)^{\beta_k} = \sum_{k=1}^{n} \beta_k \left(\frac{\theta}{2} + \frac{\theta_k}{2}\right) = \theta + c_k''$



$\Rightarrow \arg(w-w_k) = 2\pi - (\pi - \theta_k) - (\pi - \frac{\theta}{2} + \frac{\theta_k}{2}) = \frac{\theta}{2} + \frac{\theta_k}{2}$

$\Rightarrow \arg(B(w))$ is constant on each $\gamma_k \subset \partial D$

By continuity, $\arg(B(w)) = C \quad \forall |w| = 1$

Since $\log B(w)$ is analytic on $\overline{D}$, Cauchy integral formal

$\Rightarrow \arg B(0) = C$

Consider $\exp(i \log B(w)) = e^{-\arg B(w)} e^{i \log|B(w)|}$

$\max_{\overline{D}} \exp|i \log B(w)| \leq C = \exp(i \log B(0))$

$\Rightarrow$ it is a constant function

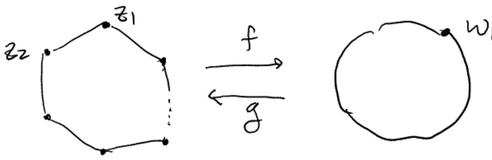
$\Rightarrow$ So is $B(w)$, $\quad B(w) = g'(w) \prod_{k=1}^{n} (w-w_k)^{+\beta_k} = C_1$

v) Thus, $g'(w) = C_1 \prod_k (w - w_k)^{-\beta_k}$

$$\Rightarrow g(w) = C_1 \int \prod_k (w-w_k)^{-\beta_k}\, dw + C_2 \; : \text{ the conformal map from disk to polygon}$$

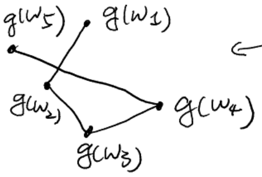
(the Schwarz–Christoffel formula)

<u>rmk</u> it is not that precise



$w_k = f(z_k)$: determined by f

Say it differently, if we just choose $\{w_k\} \subset \partial D$ and $\{\beta_k\}$ with $\sum \beta_k = 2$, then $g(\partial D)$ might be



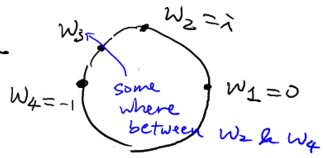
$\leftarrow$ if $g(\partial D)$ does not have self-intersection, $g : \overline{D} \to \overline{P}$: conformal equivalence

(proof uses argument principle)

§2 rectangle [Ahlfors, §2.3 of ch. 6]

i) By Aut(D), we can arrange the position of w_1, w_2, w_3 $\boxed{\text{Exercise}}$

Focus on the rectangle case, we may assume



ii) D is conformally equivalent to $\mathbb{H}$

By the similar argument $g(w) = \int \prod_{k=1}^{n-1} (w - \xi_k)^{-\beta_k}\, dw \; : \mathbb{H} \to \text{polygon}$

$\xi_k \in \mathbb{R} = \partial \mathbb{H}$

(assume $f(w_n) = \infty$)

In the rectangle case,

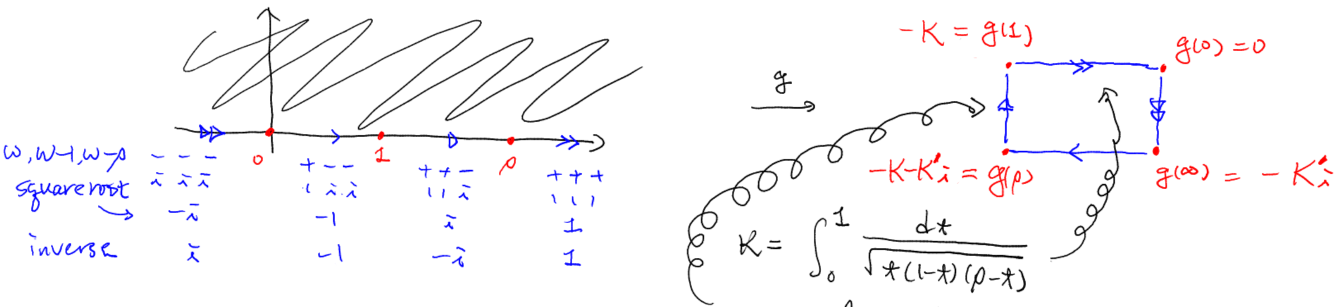
$\beta_k = \frac{1}{2}$, and we may assume $(\xi_1, \xi_2, \xi_3) = (0, 1, \rho \in (1, \infty))$

iii) $$g(w) = \int_0^w \frac{dw}{\sqrt{w(w-1)(w-\rho)}} \quad \text{for } w \in \overline{\mathbb{H}}$$

(this is an elliptic integral)

Take the branch so that $\sqrt{w}, \sqrt{w-1}, \sqrt{w-\rho} \in$ first quadrant for $w \in \overline{\mathbb{H}}$

(we will see why immediately)



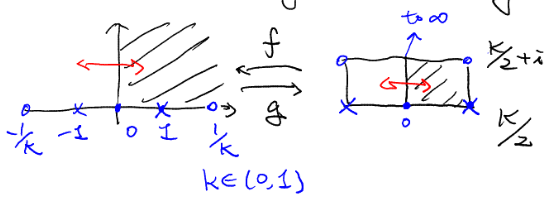
$$K = \int_0^1 \frac{dt}{\sqrt{t(1-t)(\rho - t)}}$$

$$K' = \int_1^\rho \frac{dt}{\sqrt{t(t-1)(\rho-t)}}$$

about $\twoheadrightarrow$: $K = \int_\rho^\infty \frac{dt}{\sqrt{t(t-1)(t-\rho)}}$ $\quad t = \frac{\rho}{s}$

about $\twoheadrightarrow$: $K' = \int_0^\infty \frac{dt}{\sqrt{t(t+1)(t+\rho)}}$ (one can find the similar change of variable)

iv) more symmetric way:

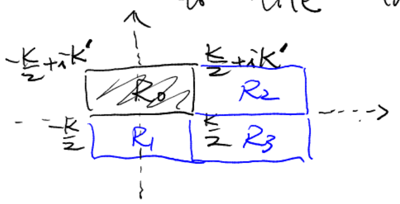


$$\Rightarrow g(w) = \int_0^w \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}$$

from $\left((w-1)(w+1)(w-\frac{1}{k})(w+\frac{1}{k})\right)^{-\frac{1}{2}}$

where
$$K = \int_{-1}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$
$$K' = \int_1^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

v) in any case, we can apply the reflection principle on f to tile the whole $\mathbb{C}$



for $z \in R_1 \Rightarrow f(z) = \overline{f(\bar{z})}$

for $z \in R_2 \Rightarrow f(z) = \overline{f(K - \bar{z})}$

↳ reflection along $x = \frac{K}{2}$

for $z \in R_3 \Rightarrow \cdots$

$\Rightarrow$ It is not hard to see that the extension is doubly periodic with period $2K$ & $2K'i$: $f(z+2K) = f(z) = f(z+2K'i)$

(elliptic function)

e.g.

$y = iK'$

$z \in R_1$

$z + 2iK' \in R_1 + 2iK'$

↳ reflect along $y = iK'$

$\overline{z + 2iK' - iK'} + iK' = \bar{z}$

$f(z+2iK') = \overline{f(\bar{z})}$ = that on R_1

§3 more on harmonic functions [Ahlfors, §3 of ch. 6]

goal: use Dirichlet problem to study the conformal equivalence between multiply connected regions

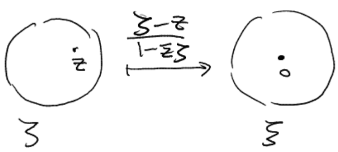
i) Poisson formula

D: unit disk u: harmonic on D, continuous on $\bar{D}$

$$\Rightarrow u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, d\theta$$

[By constructing its conjugate harmonic and Cauchy integral formula]

$u(z) = ?$



$$\tilde{u}(\xi) = u\left(\frac{\xi + z}{1+\bar{z}\xi}\right)$$

$$u(z) = \tilde{u}(0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(e^{i\phi})\, d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u\left(\frac{e^{i\phi}+z}{1+\bar{z}e^{i\phi}}\right) d\phi$$

let $e^{i\theta} = \dfrac{e^{i\phi}+z}{1+\bar{z}e^{i\phi}} \Rightarrow e^{i\phi} = \dfrac{e^{i\theta}-z}{1-\bar{z}e^{i\theta}}$

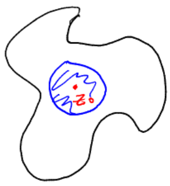
$$e^{i\phi}\,d\phi = \left(\frac{e^{i\theta}}{1-\bar{z}e^{i\theta}} + \frac{e^{i\theta}\bar{z}(e^{i\theta}-z)}{(1-\bar{z}e^{i\theta})^2}\right)d\theta$$

$$\Rightarrow d\phi = \left(\frac{e^{i\theta}}{e^{i\theta}-z} + \frac{\bar{z}}{e^{-i\theta}-\bar{z}}\right)d\theta = \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\theta$$

$$\Rightarrow u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,u(e^{i\theta})\,d\theta$$

$\boxed{\text{check}}$ Given $u \in C^0(\partial D)$, the above formula defines a harmonic function on D, with $\lim\limits_{z\to e^{i\theta}} u(z) = u(e^{i\theta})$

ii) mean-value property



$u \in C^0(\Omega)$ is said to obey the mean value property if $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta$ if $\overline{B(z_0;r)} \subset \Omega$

lemma u has no maximum / minimum unless $u \equiv$ constant

pf: if $\max\limits_{\Omega} u = u(z_0)$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} u(z_0)\,d\theta = u(z_0)$$

$\Rightarrow u(z_0+re^{i\theta}) = u(z_0)$ ... ※

thm u is necessarily harmonic (note that NO C^2-assumption is imposed here)

pf: Choose any $\overline{B(z_0;r)} \subset \Omega$.

Poisson formula produces v on $\overline{B(z_0;r)}$ such that $\begin{cases} \Delta v = 0 & \text{on } B(z_0;r) \\ v = u & \text{on } \partial B(z_0;r) \end{cases}$

$\Rightarrow$ $u - v = 0$ on $\partial B(z_0;r)$, satisfies the mean-value property on $B(z_0;r)$

By the max/min principle, $u - v \equiv 0$ on $B(z_0;r)$ ※

iii) Harnack's principle

By Poisson, $u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2-|z|^2}{|\rho e^{i\theta}-z|^2}\,u(\rho e^{i\theta})\,d\theta$

Both $u(0)$ & $u(z)$ are determines by $u(\rho e^{i\theta})$

relation ?

Denote $|z|$ by r,

$$\rho - r \le |\rho e^{i\theta} - z| \le \rho + r$$

$$\Rightarrow \frac{\rho-r}{\rho+r} = \frac{\rho^2-r^2}{(\rho+r)^2} \le \frac{\rho^2-r^2}{|\rho e^{i\theta}-z|^2} \le \frac{\rho^2-r^2}{(\rho-r)^2} = \frac{\rho+r}{\rho-r}$$

Hence, $\frac{\rho-r}{\rho+r}u(0) \le u(z) \le \frac{\rho+r}{\rho-r}u(0)$ (the Harnack inequality)

provided $u(\rho e^{i\theta}) \ge 0$ ($\Rightarrow u(z) \ge 0 \ \forall z \in B(0;\rho)$)

thm $\Omega_n \to \Omega$ (exhaustion)

$u_n(z)$: harmonic on Ω_n $\quad$ monotone increasing

$\forall z \in \Omega$ if $u_n(z) \le u_{n+1}(z)$ for $n \ge N(z)$

Then, either $u_n(z) \to \infty$ $\quad$ uniformly on

or $u(z)$: harmonic on Ω $\quad$ any compact subset

pf: If $\lim_{n\to\infty} u_n(z_0) = \infty$ for some $z_0 \in \Omega$

choose $r > 0$ such that $\overline{B(z_0; 2r)} \subset \Omega$

By Harnack $\Rightarrow$ $u_n(z) - u_{n_0}(z) \ge \frac{1}{3}(u_n(z_0) - u_{n_0}(z_0))$

or $u_n(z) \ge \frac{1}{3} u_n(z_0) + (u_{n_0}(z) - \frac{1}{3} u_{n_0}(z_0))$ $\quad \forall\, n \ge n_0$, $z \in \overline{B(z_0; r)}$

$\Rightarrow u_n(z) \to \infty$ uniformly on $\overline{B(z_0; r)}$

If $u(z_0)$ is finite, choose r so that $\overline{B(z_0; 2r)} \subset \Omega$

again by Harnack, $0 \le u_n(z) - u_m(z) \le 3(u_n(z_0) - u_m(z_0))$

$\forall\, n \ge m$, $z \in \overline{B(z_0; r)}$

$\Rightarrow$ convergence is controlled by the convergence at z_0

hence uniform on $\overline{B(z_0; r)}$ ※