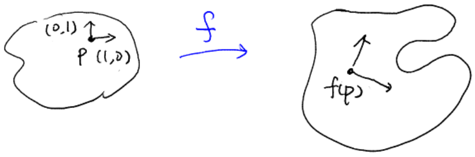
XI Riemann Mapping Theorem

§1 conformal mapping

a map $f : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ is called conformal if it preserves the angles



$f = (u, v)$

$(1,0) \mapsto (u_x, v_x)$

$(0,1) \mapsto (u_y, v_y)$

shall be linearly independent

We may assume $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} > 0$

Let $A = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$, the conformal condition meas that

$$\frac{\langle s, t\rangle}{|s||t|} = \frac{\langle As, At\rangle}{|As|\ |At|} \quad \forall s, t \in \mathbb{R}^2 \quad \boxed{\text{staright forward}} \Rightarrow u_x = v_y,\ u_y = -v_x$$

Therefore, $f = u + iv$ is analytic

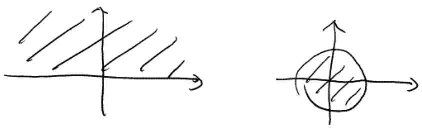
$\boxed{Q}$ $\mathcal{U}, \mathcal{V} \subset \mathbb{C}$, when are they conformal equivalent?

Namely, does there exist a bijective analytic map between $\mathcal{U}$ & $\mathcal{V}$?

<u>examples</u> 1) $\mathcal{U} = \mathbb{C}$, $\mathcal{V} = \mathbb{D}$ $\quad \mathcal{U} \xrightarrow{f} \mathcal{V}$?

By Liouville, f must be a constant

2) $\mathcal{U} = \mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \} \xrightarrow{f} \mathcal{V} = \mathbb{D}$



Regard them as a subset of $\hat{\mathbb{C}}$

$f : i \mapsto 0$

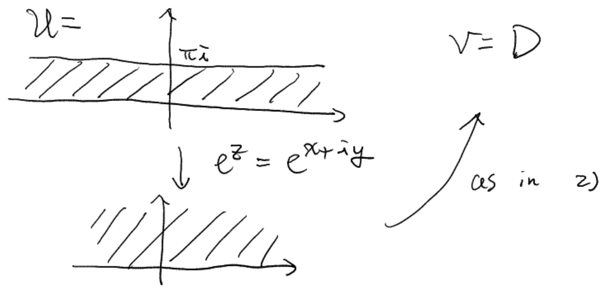
$0 \mapsto -1$

$\infty \mapsto 1$

Take $f(z) = \dfrac{z - i}{z + i}$

$\boxed{\text{check}}$ f defines a conformal equivalence

3)



$\mathcal{V} = \mathbb{D}$

take $f(z) = \dfrac{e^z - i}{e^z + i}$

4) polygon regions ?

later ...

another viewpoint : analytic functions on $\mathcal{U}$

- $\mathcal{U} = \mathbb{C}$: a lot, but NO bounded ones except constants
- $\mathcal{U} = \mathbb{D}$: a lot, also a lot bounded ones

(• $\mathcal{U} = \hat{\mathbb{C}}$: only constants $\boxed{\text{HW 9}}$)

§2 Riemann mapping theorem [Ahlfors, §1.1 of ch. 6]

thm $\Omega \subsetneq \mathbb{C}$, open and simply connected. Then, for any $z_0 \in \Omega$ $\exists!\, f : \Omega \to D$. which satisfies $f(z_0) = 0$, $f'(z_0) > 0$. and which gives a conformal equivalence.

recall $\mathrm{Aut}(D) = \{ e^{i\theta} \frac{z-a}{1-\bar{a}z} \mid a \in D,\ e^{i\theta} \in S^1 \}$ by Schwarz lemma

$a \leftrightarrow f(z_0)$, $e^{i\theta} \leftrightarrow f'(z_0)$

[$f'(z_0) > 0$ means $f'(z_0) \in \mathbb{R}_{>0}$]

It is easy to argue uniqueness from it

recall Ω: simply-connected means that $n(\gamma, a) = 0$ $\forall \gamma \subset \Omega$, $a \notin \Omega$

$\Rightarrow \int_\gamma f\, dz = 0$ $\forall f \in \mathcal{H}(\Omega)$, and closed curve $\gamma \subset \Omega$

$\Rightarrow$ any nowhere zero analytic function f has a well-defined log, square root, etc.

pf: Consider $\mathcal{F} = \{ g \in \mathcal{H}(\Omega) \mid g:\text{injective},\ g(z_0) = 0,\ g'(z_0) > 0,\ |g(z)| < 1 \}$

Due to Schwarz lemma, we are looking for $f \in \mathcal{F}$ with maximal $f'(z_0)$

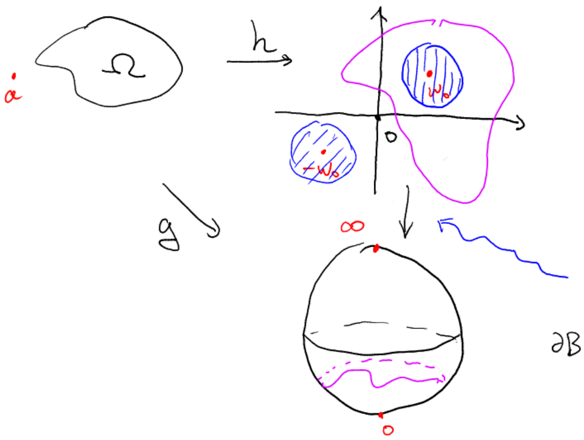
[Suppose true: consider $D \xrightarrow{g^{-1}} \Omega \xrightarrow{f \in \mathcal{F}} D$]
$0 \longmapsto z_0 \longmapsto 0$

1° $\mathcal{F} \neq \phi$. By assumption, $\exists\, a \in \mathbb{C} \setminus \Omega$.

Since $z - a \neq 0$ on Ω, $h(z) = \sqrt{z-a}$ is well-defined on Ω

[intuitively, the image belongs to some half-plane we can apply example 2 to map it into D]

Denote $h(z_0)$ by w_0. By open mapping, $h(\Omega) \supset B(w_0; \rho)$



Since both $z-a$ and $h(z) = \sqrt{z-a}$ are injective, $B(-w_0; \rho) \subset \mathbb{C} \setminus h(\Omega)$

As a consequence, $|w_0| > \rho \geq \frac{\rho}{2}$

$-w_0 \mapsto \infty$
$w_0 \mapsto 0$
$\partial B(-w_0; \rho) \mapsto$ inside D

also, normalize it so that $g'(z_0) > 0$

$w = -w_0 + \rho e^{i\theta}$

$$\frac{w - w_0}{w + w_0} = \frac{1}{\rho}(-2w_0 + \rho e^{i\theta})$$

$$\left|\frac{1}{\rho}(-2w_0 + \rho e^{i\theta})\right| < \frac{4|w_0|}{\rho}$$

$$\frac{\partial}{\partial w}\Big|_{w = w_0} \left(\frac{w - w_0}{w + w_0}\right) = + \frac{2}{w_0}$$

$$\Rightarrow g(w) = \frac{\rho}{4|h(z_0)|} \frac{h(z_0)}{|h(z_0)|} \frac{|h'(z_0)|}{h'(z_0)} \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \in \mathcal{F}$$

2° Let $B = \sup\{ g'(z_0) \mid g \in \mathcal{F} \} \in \mathbb{R}_{>0} \cup \{\infty\}$

$\exists f_n \in \mathcal{F}$ such that $f_n'(z_0) \to B$ as $n \to \infty$

By Montel, we may assume f_n converges to $f \in \mathcal{H}(\Omega)$ uniformly on compact subset of Ω.

As a consequence, $B < \infty$

3° [injectivity of f] At first, it follows from $f'(z_0) = B > 0$ that f is not a constant function

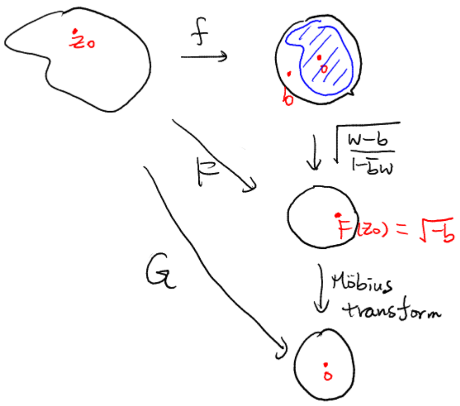
Choose any $z_1 \in \Omega$. $f(z) - f(z_1) = 0 \overset{?}{\Rightarrow} z = z_1$

$g_n(z) = f_n(z) - f_n(z_1) \longrightarrow g(z) = f(z) - f(z_1)$

uniformly on any compact subset of $\Omega \setminus \{z_1\}$

But $g_n(z)$ = nowhere zero $\Rightarrow$ $g(z)$: nowhere zero, or identically zero ← cannot happen

4° [surjectivity of f] If $\exists b \in D \setminus f(\Omega)$, what happens?



Try to use the extremal property of $f'(z_0)$

Compose with $\frac{w-b}{1-\bar{b}w}$ ⇝ square root ⇝ adjust $z_0 \mapsto 0$

$F(z) = \sqrt{\frac{f(z)-b}{1-\bar{b}f(z)}} : \Omega \to D$. injective

$G(z) = e^{i\theta} \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)}$ choose $e^{i\theta}$ so that $G'(z_0) > 0$

corresponding derivatives $f'(z_0) = B$

$$\left(\frac{w-b}{1-\bar{b}w}\right)^{\frac{1}{2}} \rightsquigarrow \frac{1}{2}\left(\frac{w-b}{1-\bar{b}w}\right)^{-\frac{1}{2}}\left(\frac{1}{1-\bar{b}w} + \frac{(w-b)\bar{b}}{(1-\bar{b}w)^2}\right) \quad @\ w=0 \quad = \frac{1}{2}\frac{1}{\sqrt{-b}}(1-|b|^2)$$

$$e^{i\theta}\left(\frac{\zeta - F(z_0)}{1-\overline{F(z_0)}\zeta}\right) \rightsquigarrow e^{i\theta}\left(\frac{1}{1-\overline{F(z_0)}\zeta} + \frac{(\zeta - F(z_0))\overline{F(z_0)}}{(1-\overline{F(z_0)}\zeta)^2}\right) \quad @\ \zeta = F(z_0) \quad = e^{i\theta}\frac{1}{1-|b|}$$

$$\Rightarrow G'(0) = \frac{1+|b|}{2\sqrt{|b|}} B > B \quad \Rightarrow\Leftarrow$$

[intuitively, $\sqrt{\ }$ is an expanding map.
pretend that Schwarz lemma applies to $f \circ G^{-1}$]

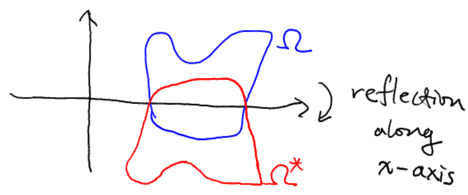
§3 Schwarz reflection principle

goal understand the boundary behavior of a conformal equivalence

When could $f: \mathcal{U} \to \mathcal{V}$ be extened to the boundary, or across the boundary?

toy model / observation: $\Omega \subset \mathbb{C}$ open & connected

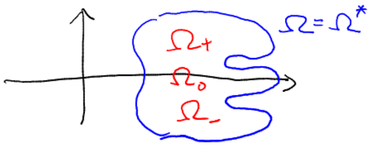
$\Omega^* = \{ z \in \mathbb{C} \mid \bar{z} \in \Omega \}$



$f : \Omega \to \mathbb{C}$ analytic $\Leftrightarrow$ $\frac{\partial}{\partial \bar z} f(z) = 0$

$\rightsquigarrow$ define $g : \Omega^* \to \mathbb{C}$ by $\overline{f(\bar z)}$ $\qquad \frac{\partial}{\partial \bar z} g(z) = \frac{\partial}{\partial \bar z} \overline{f(\bar z)} = \overline{\frac{\partial}{\partial z} f(\bar z)} = 0$ $\boxed{\text{check}}$

In particular, if $\Omega = \Omega^*$, we have two analytic functions on it.
Could they be the same?



If so, $\overline{f(\bar z)} = f(z)$

When $z \in \Omega \cap \mathbb{R}$, $\overline{f(z)} = f(z) \Leftrightarrow f$: real-valued

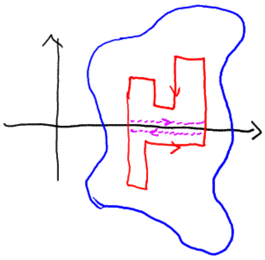
<u>thm</u> (Schwarz reflection principle) Suppose that $\Omega = \Omega^*$

If $f : \Omega_+ \cup \Omega_0 \to \mathbb{C}$ is continuous, analytic on Ω_+, and takes real value on Ω_0.

then, $\exists\, F$: analytic on Ω which extends f.

pf: For $z \in \Omega_-$ define $F(z)$ by $\overline{f(\bar z)}$.

$\Rightarrow F \in \mathcal{C}(\Omega)$, and analytic on $\Omega_+ \cup \Omega_-$



It is not hard to justify the criterion of Morera:

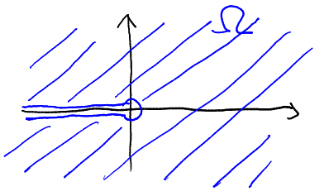
$\int_\gamma F\, dz = 0$ for any closed, rectangular loop $\divideontimes$

§4 boundary behavior

[Ahlfors, §1.2 and 1.3 of ch. 6]
[Stein, §4.3 of ch. 8]

Start with a bad example.

<u>example</u> $\Omega = \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \le 0\} \xrightarrow{f} D$



$z \longmapsto \dfrac{\sqrt{z} + 1}{\sqrt{z} - 1}$

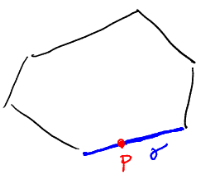
$\left(\dfrac{w+1}{w-1}\right)^2 \longleftarrow w$

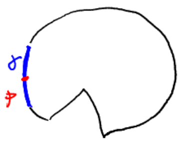
$\partial \Omega = \{x \in \mathbb{R} \mid x \le 0\}$

cannot really extend f

but okay for f^{-1} (except $z = \infty \leftrightarrow w = 1$)

For simplicity, consider only the simplest boundary:





$\gamma \subset \partial \Omega$: a line segment, or part of a circle

and it is one-sided.

Namely. $\forall\, p \in \gamma$, $\exists\, B(p;\varepsilon)$ such that

$B(p;\varepsilon) \setminus \gamma$ has two components, and

$\Omega \cap B(p;\varepsilon)$ is exactly one of them

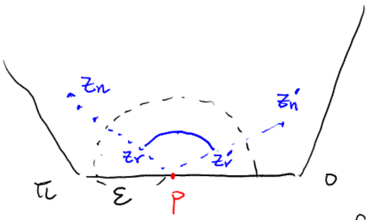
goal/assumption: $f: \Omega \to D$ (or $D \to \Omega$: bounded) extend it to $\gamma \subset \partial\Omega$
conformal equivalence

1° area : $f = (u,v)$ $\quad J(f) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} = u_x^2 + v_x^2 = |f'(z)|^2$

$\mathcal{U} \subset \Omega \qquad \text{Area}(f(\mathcal{U})) = \iint_{\mathcal{U}} |f'(z)|^2 \, dx\, dy$

2°

(not only $f(z_n)$ converges, but also $f(z_n')$ converges to the same limit)



<u>lemma</u> $\forall\ 0 < r < \varepsilon$, given $z_r, z_r' \in \Omega$, $|z_r - p| = |z_r' - p| = r$

Then, $\exists\ r_n \to 0$ such that $\lim_{n\to\infty} |f(z_n) - f(z_n')| = 0$

pf: Denote $|f(z_r) - f(z_r')|$ by $\rho(r)$

Suppose the assertion is NOT true. Then, $\rho(r) \geq \delta > 0$ $\quad \forall\ 0 < r < \varepsilon$

$\rho(r) \leq \int_\alpha |f'(\zeta)|\,|d\zeta| \qquad \alpha$: any arc in Ω connecting z_r & z_r'

take $\alpha = p + re^{i\theta}$ $\quad r$: fixed. θ: parameter

$z_r = p + re^{i\theta_1(r)}$
$z_r' = p + re^{i\theta_2(r)}$

$\Rightarrow \quad \delta \leq \int_{\theta_1(r)}^{\theta_2(r)} |f'(z)|\, r\, d\theta$

[try to relate to area] $\leq \left(\int_{\theta_1(r)}^{\theta_2(r)} |f'(z)|^2\, r\, d\theta\right)^{\frac{1}{2}} \left(\int_{\theta_1(r)}^{\theta_2(r)} r\, d\theta\right)^{\frac{1}{2}}$

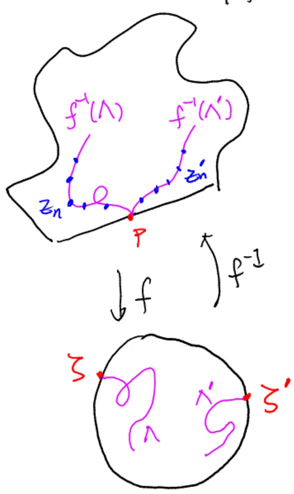
[Cauchy-Schwarz on $|f'(z)|r^{\frac{1}{2}}$ & $r^{\frac{1}{2}}$]

$\Rightarrow \quad \frac{\delta^2}{r} \leq \pi \int_0^\pi |f'(z)|^2\, r\, d\theta$

$\Rightarrow \quad \delta^2 \int_0^\varepsilon \frac{1}{r}\, dr \leq \pi \iint_0^\varepsilon{}_0^\pi |f'(z)|^2\, r\, dr\, d\theta = < \infty \qquad \rightarrow\leftarrow$

3° <u>lemma</u> With the same setting, $\lim_{z\to p} f(z)$ exists.
$z \in \Omega$

pf: Let $z_n \in \Omega$, $z_n \to p$.



Since $f(\Omega)$ is bounded, we may assume $\lim_{n\to\infty} f(z_n)$ exists

By $\boxed{\text{Homework}}$ the limit belongs to ∂D

Suppose the lemma is false. $\exists\ \{z_n\} \to p$. $\{z_n'\} \to p$.
such that $\lim f(z_n) = \zeta \neq \zeta' = \lim f(z_n')$

Then, we can easily construct z_r & z_r' violate the previous lemma :

Λ, Λ' : curves in D such that $f(z_n) \in \Lambda$, $f(z_n') \in \Lambda'$
(basically, joining $f(z_n)$ together)

Since $\zeta \neq \zeta'$, $|w - w'| > \delta > 0 \quad \forall\ w \in \Lambda,\ w' \in \Lambda'$

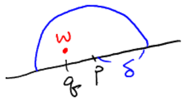
But the points on $f^{-1}(\Lambda)$ & $f^{-1}(\Lambda')$ would violate the lemma #

4° Now, we know that $\lim_{z\to p} f(z)$ exists

We can take the limit $\forall\ p \in$ the boundary segment $\gamma \subset \partial\Omega$

claim the extension is continuous on $\Omega \cup \gamma$

pf: given $\varepsilon > 0$ $\exists \delta > 0$ such that $|f(z) - f(p)| < \varepsilon$ $\forall z \in \Omega$, $|z-p| < \delta$

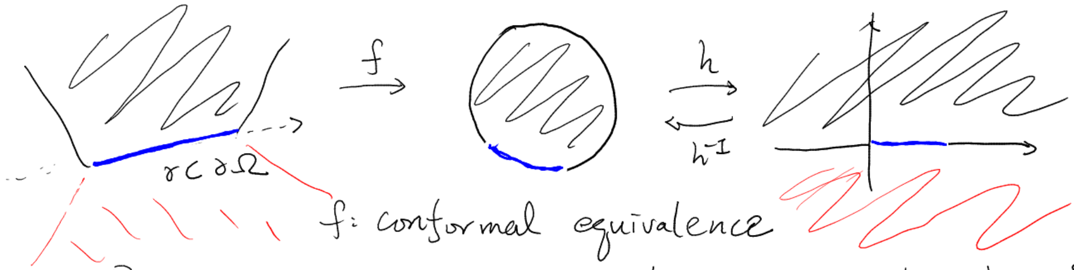


For any $q \in \partial\Omega$, $|q-p| < \delta$

Since $\lim_{z \to q} f(z) = f(q)$. $\exists\, w \in B(p,\delta) \cap \Omega$ such that $|f(w) - f(q)| < \varepsilon$

$$\Rightarrow |f(p) - f(q)| < |f(p) - f(w)| + |f(w) - f(q)| < 2\varepsilon \quad \#$$

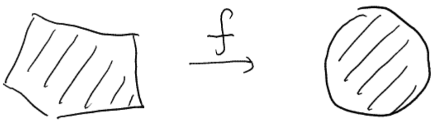
5°



f: conformal equivalence

By combining the above discussion with the Schwarz reflection principle, f can be extended to an analytic function across γ

6°



how to construct the conformal equivalence between a polygon and the disk?

We have to study more about the behavior near the vertex [NEXT WEEK].