X. normal family [Ahlfors, §5 of ch. 5]

$\{f_n\}$: sequence of (analytic) functions defined on $\Omega \underset{\text{open}}{\subset} \mathbb{C}$

Under what condition would there be a subsequence converge uniformly on any compact subset of Ω?

[analytic $\overset{\text{Cauchy}}{\Longrightarrow}$ f' is controlled by f (in a larger region)
some suitable condition on $\{f_n\}$ shall be enough]

§1 normal and compactness

$\mathcal{F}$: family of functions defined on Ω

<u>defn</u> $\mathcal{F}$ is equicontinuous on E if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ for any $|z-w| < \delta$ and $z, w \in E$

<u>defn</u> $\mathcal{F}$ is said to be normal if any $\{f_n\} \subset \mathcal{F}$ has a subsequence which converges uniformly on any compact subset of Ω (the limit is not required to be in $\mathcal{F}$)

describe by a "norm"?

Since the domain is not compact, we cannot simply consider the sup-norm

<u>idea</u> i) exhaust Ω by a nested compact subsets $\{K_n\}$
ii) consider the "weighted" sum of the sup-norms.
iii) the norm governs the desired convergence property.

for i) $\exists$ K_j : compact sets. such that
$$\begin{cases} K_j \subset \text{interior}(K_{j+1}) \\ \Omega = \bigcup_j K_j, \quad \text{if } K : \text{compact} \subset \Omega \Rightarrow K \subset K_j \text{ for some } j \end{cases}$$

$\boxed{\text{TA section}}$

for ii) for $f, g \in \mathcal{F} \subset \mathcal{C}(\Omega; \mathbb{C})$ $\quad d_j(f,g) = \sup_{K_j} |f-g|$

deform it by $\delta_j(f,g) = \dfrac{d_j(f,g)}{1 + d_j(f,g)}$ (no longer linear
not a normed vector space
but can still examine the topology)

$$\begin{cases} \delta_j(f,g) < 1 \\ \delta_j(f,g) \le d_j(f,g) \text{ : straightforward} \\ \text{When } d_j \le 1 \iff \delta_j \le \frac{1}{2}, \quad \delta_j \ge \frac{1}{2} d_j \end{cases}$$

(behavior of $\frac{x}{1+x}$)

for iii) define $\rho(f,g) = \sum_{j=1}^{\infty} \dfrac{\delta_j(f,g)}{2^j}$

<u>prop</u> $f_n \to f$ in the ρ distance if and only if
$f_n \to f$ uniformly on any compact subsets of Ω

pf: $\Rightarrow$) $K \subset K_j$ for some j: fixed

Given $\varepsilon > 0$. $\exists N$ such that $\rho(f_n, f) < \varepsilon$ $\forall n \geq N$

$\Rightarrow \delta_j(f_n, f) < 2^j \varepsilon$

$\Rightarrow f_n \to f$ in sup-norm on K_j $\Leftrightarrow$ uniform convergence on K_j

$\Leftarrow$) if $f_n \to f$ uniformly on each K_j

$\forall \varepsilon > 0$, $\exists J$ such that $2^{-J} < \varepsilon/2$

$1 = \underbrace{\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^j}}_{\text{less than } 1} + \underbrace{\frac{1}{2^{j+1}} + \cdots}_{\text{less than } 2^{-j}}$

for $1 \leq j \leq J$, $\exists N$

such that $\delta_j(f_n, f) < \frac{\varepsilon}{2J}$

Hence, $\rho(f_n, f) < \frac{\varepsilon}{2J} \cdot J + \frac{\varepsilon}{2} = \varepsilon$ ※

Cor $(\mathcal{C}(\Omega; G), \rho)$ is a complete metric space.

pf: $\{f_n\}$: Cauchy $\Rightarrow$ $\exists$ uniform convergent subsequence on $K_1, \cdots K_j \cdots$

$\Rightarrow$ choose diagonal subsequence $\cdots$ ※

defn $\mathcal{F} \subset \mathcal{C}(\Omega; \mathbb{C})$ is normal if any sequence of $\mathcal{F}$ has a subsequence that converges to some $f \in \mathcal{C}(\Omega; \mathbb{C})$ wrt ρ

equivalently. $\bar{\mathcal{F}}$ is compact (Balzano-Weierstrass)

closure

Goal find simple / minimal condition to characterize normality in particular, for analytic functions.

§2 totally bounded and Arzela's theorem

totally bounded $\mathcal{F} \subset (\mathcal{C}(\Omega), \rho)$ is totally bounded if $\forall \varepsilon > 0$, $\mathcal{F}$ can be covered by finite ε-balls.

prop $\mathcal{F}$ is normal if and only if it is totally bounded

pf: $\Rightarrow$) $\bar{\mathcal{F}}$: compact $\Rightarrow$ totally bounded $\Rightarrow$ so is $\mathcal{F}$

$\Leftarrow$) balls cover $\mathcal{F}$ must cover $\bar{\mathcal{F}}$ $\Rightarrow \cdots \Rightarrow$ any open cover of $\bar{\mathcal{F}}$ admits a finite subcover ※

prop $\mathcal{F}$ is totally bounded if and only if $\forall$ compact set $K \subset \Omega$ and $\varepsilon > 0$, $\exists \{f_1, \ldots, f_n\} \subset \mathcal{F}$ such that for any $f \in \mathcal{F}$, $\exists f_\ell$ so that $\sup_K |f - f_\ell| < \varepsilon$

pf: $\Rightarrow$) $K \subset K_J$ for some J

cover $\mathcal{F}$ by finite $2^{-(J+1)}\varepsilon$ balls $\Rightarrow \sum_j 2^j \delta_j(f_\ell, f) < 2^{-(J+1)}\varepsilon$ for some f_ℓ

$\Rightarrow \delta_J(f_\ell, f) < \frac{\varepsilon}{2} \xRightarrow{\text{say, } \varepsilon \leq 1} d_J(f_\ell, f) < \varepsilon$

$\Leftarrow$) Given $\varepsilon > 0$. $\exists J$ such that $2^{-J} < \frac{\varepsilon}{2}$

For K_J & $\frac{\varepsilon}{2J}$, $\mathcal{F}$ can be covered by finite $\frac{\varepsilon}{2J}$-balls (wrt sup-norm over K_J)

Namely, $\exists \{f_1, \cdots, f_n\}$ such that for any f, $\exists f_\ell$ so that $d_J(f_\ell, f) < \frac{\varepsilon}{2J}$

But $d_j < d_J$ for $j \le J$ $\Rightarrow$ $\rho(f_\ell, f) < \frac{\varepsilon}{2J} \cdot J + \frac{\varepsilon}{2}$ ※

<u>theorem</u> (Arzela) $\mathcal{F} \subset \mathcal{C}(\Omega)$ is normal if and only if

(i) $\mathcal{F}$ is equi-continuous on any compact set $K \subset \Omega$

(ii) for any $z \in \Omega$, $\{f(z) \mid f \in \mathcal{F}\}$ is bounded

pf: normal $\Leftrightarrow$ any sequence admits a uniform convergent subsequence

in the sense of uniform convergent on any compact subset

$\Rightarrow$ By the usual Arzela-Ascoli.

$\Leftarrow$ Given a sequence, $\exists$ convergent subsequence over K_1

$\cup$ $\exists$ $\cdots$ over K_2

Then, the diagonal argument would finish the proof ※

§3 family of analytic functions

<u>recall</u> (Weierstrass) f_n: analytic $\to f$ uniformly on compact subsets

then f: analytic and $f_n' \to f$ $\cdots$

<u>Q</u> $\mathcal{F}$: analytic on Ω, when is it normal?

<u>discussion</u> if so, Arzela $\Rightarrow$ equi-continuous on any compact subset

and uniform bounded $\forall z$

$\Rightarrow$ uniform bounded on any compact subset

<u>defn</u> $\mathcal{F}$ is called locally bounded if it is uniform bounded on any compact subset, equivalently, $\forall p \in \Omega$, there exists a neighborhood so that $\mathcal{F}$ is uniform bounded there

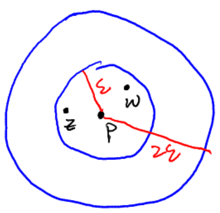
<u>theorem</u> (Montel) $\mathcal{F}$: analytic is normal if and only if it is locally bounded

pf: $\Rightarrow$) By Arzela as above

($\Leftarrow$) It remains to show $\forall p \in \Omega$, $\exists$ neighborhood so that $\mathcal{F}$ is equicontinuous there.

To start, $\exists B(p, 3\varepsilon) \subset \Omega$ and $M > 0$

so that $|f(z)| < M$ $\forall z \in B(p, 3\varepsilon)$ and $f \in \mathcal{F}$



For $z, w \in B(p, \varepsilon)$

$$f(z) - f(w) = \frac{1}{2\pi i} \int_{|\zeta - p| = 2\varepsilon} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) f(\zeta)\, d\zeta$$

$$\Rightarrow \quad |f(z)-f(w)| \leq \frac{1}{2\pi}\frac{M}{\varepsilon^2} 4\pi\varepsilon\, |z-w| = \frac{2M}{\varepsilon}|z-w|$$ ※

With the convergence theorem of Weierstrass, $\mathcal{F}'$ (derivative of $\mathcal{F}$) shall also be a normal family.

<u>Cor</u> locally bounded family of analytic function has locally bounded derivative

Pf: $f'(z) = \frac{1}{2\pi i}\int_{|\zeta-p|=2\varepsilon} \frac{f(\zeta)}{(\zeta-z)^2}d\zeta \Rightarrow |f'(z)| \leq \frac{1}{2\pi}\frac{M}{\varepsilon^2}4\pi\varepsilon = \frac{2M}{\varepsilon}$ ※

§4 meromorphic functions

How about meromorphic functions?

Include ∞ in the target space $\rightsquigarrow \hat{\mathbb{C}} \cong S^2$ stereographic projection

Endow S^2 the metric from $\mathbb{R}^3$

(x_1, x_2, x_3), $(y_1, y_2, y_3) \rightsquigarrow$ distance$^2 = \sum_{j=1}^{3}(x_j - y_j)^2$

In stereographic projection $z_1, z_2 \in \mathbb{C}$

$$\updownarrow$$

$$\left(\frac{2z_1}{1+|z_1|^2}, \frac{1-|z_1|^2}{1+|z_1|^2}\right), \left(\frac{2z_2}{1+|z_2|^2}, \frac{1-|z_2|^2}{1+|z_2|^2}\right) \in \mathbb{C}\times\mathbb{R} \cong \mathbb{R}^3$$

direct computation

$$d_{\hat{\mathbb{C}}}(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$$

$$d_{\hat{\mathbb{C}}}(z_1, \infty) = \frac{2}{\sqrt{1+|z_1|^2}} \qquad d_{\hat{\mathbb{C}}}(0, z_2) = \frac{2|z_2|}{\sqrt{1+|z_2|^2}}$$

reciprocal $z_1 = 1/z_2$

Consider $\Omega \overset{\text{open}}{\subset} \mathbb{C} \xrightarrow{\text{continuous}} (\hat{\mathbb{C}}, d)$: a complete, in fact compact, metric space.

- $\mathcal{C}(\Omega; \hat{\mathbb{C}})$ contains more functions: $\begin{cases} \infty\text{ - constant function} \\ \text{meromorphic functions} \end{cases}$ and more ....
- the normal family discussion still holds.

  in particular, Arzela $\begin{cases} \text{uniform bounded } \forall z \\ \text{equi-continuous on compact subsets} \end{cases}$

  target space is compact: holds naturally;

  for any $z \in \Omega$, $\{f(z) \mid f \in \mathcal{F}\}$ always has compact closure

  replace $|f(z) - f(w)|$ by $d_{\hat{\mathbb{C}}}(f(z), f(w)) = \frac{|f(z)-f(w)|}{\sqrt{1+|f(z)|^2}\sqrt{1+|f(w)|^2}}$

<u>Upshot</u> i) $\mathcal{C}(\Omega; \hat{\mathbb{C}})$ is a complete metric space

ii) $\mathcal{F} \subset \mathcal{C}(\Omega; \hat{\mathbb{C}})$ is normal if and only if it is equi-continuous on compact subsets

skip it in class

Pf: (For i)) Let us just explain that on a compact set K

Cauchy sequence $\{f_n\}$ in $d_K(f, g) = \max_{z\in K} d_{\hat{\mathbb{C}}}(f(z), g(z))$ d_K-distance

$\overset{?}{\Rightarrow} \exists f \in \mathcal{C}(K; \hat{\mathbb{C}})$ so that $f_n \to f$ uniformly on K

$\{f_n(z)\}_n$ is a Cauchy sequence in $\hat{\mathbb{C}}$ $(\forall z \in K)$

Since $(\hat{\mathbb{C}}, d_{\hat{\mathbb{C}}})$ is a compact metric space, $\lim_{n\to\infty} f_n(z)$ exists

Define $f(z)$ to be $\lim_{n\to\infty} f_n(z)$

uniform convergence:

Given $\varepsilon > 0$ $\exists N$. such that $d_K(f_n, f_m) < \varepsilon$ $\forall n, m \geq N$

For any $z \in K$, $\exists N_z \geq N$ such that $d_{\hat{\mathbb{C}}}(f_n(z), f(z)) < \varepsilon$ $\forall n \geq N_z$

$$\Rightarrow d_{\hat{\mathbb{C}}}(f_n(z), f(z)) \leq d_{\hat{\mathbb{C}}}(f_n(z), f_{N_z}(z)) + d_{\hat{\mathbb{C}}}(f_{N_z}(z), f(z)) \leq 2\varepsilon \quad \forall n \geq N$$

continuity of f

Since K is compact, $\exists \delta$ such that $d_{\hat{\mathbb{C}}}(f_N(z), f_N(w)) < \varepsilon$ for $|z-w| < \delta$, $z, w \in K$

$\left[\forall z \in K, \exists \delta_z \Rightarrow B(z; \delta_z)\text{: open cover} \Rightarrow \text{finite subcover} \Rightarrow \text{take minimal } \delta \right]$

$$d_{\hat{\mathbb{C}}}(f(z), f(w)) \leq d_{\hat{\mathbb{C}}}(f(z), f_N(z)) + d_{\hat{\mathbb{C}}}(f_N(z), f_N(w)) + d_{\hat{\mathbb{C}}}(f_N(w), f(w))$$
$$\leq 2\varepsilon + \varepsilon + 2\varepsilon = 5\varepsilon \quad \text{if } |z-w| < \delta$$

Then, use $\Omega = \bigcup_{j\geq 1} K_j$ $\rightsquigarrow$ completely the same as before

(For ii) $\Rightarrow$ $\rho = \sum_j \frac{\delta_j}{2^j} = \sum_j \frac{1}{2^j} \frac{d_j}{1+d_j}$ $d_j \to \max_{K_j} d_{\hat{\mathbb{C}}}$

$\bar{\mathcal{F}}$ is compact in the ρ-metric

Given $0 < \varepsilon < 1$, $\exists \{g_1, \cdots, g_N\} \in \mathcal{F}$ so that $\bar{\mathcal{F}} \subset \bigcup_{\ell=1}^{N} B(g_\ell; \varepsilon/3\cdot 2^{J+1})$ using the ρ-metric

$K \subset K_J$ for some J,

for any $f \in \mathcal{F}$, $\exists g_\ell$ so that $\rho(f, g_\ell) < \varepsilon/3\cdot 2^{J+1}$

$\Rightarrow \delta_J(f, g_\ell) < \varepsilon/6 < \frac{1}{2}$

$\Rightarrow d_J(f, g_\ell) < \varepsilon/3$

$\Rightarrow \max_K d_{\hat{\mathbb{C}}}(f(z), g_\ell(z)) < \varepsilon/3$

Each g_ℓ is uniform continuous on K

$\Rightarrow \exists \delta > 0$ so that $d_{\hat{\mathbb{C}}}(g_\ell(z), g_\ell(w)) < \varepsilon/3$ $\forall z, w \in K$, $|z-w| < \delta$, $1 \leq \ell \leq N$

$$\Rightarrow d_{\hat{\mathbb{C}}}(f(z), f(w)) \leq d_{\hat{\mathbb{C}}}(f(z), g_\ell(z)) + d_{\hat{\mathbb{C}}}(g_\ell(z), g_\ell(w)) + d_{\hat{\mathbb{C}}}(g_\ell(w), f(w)) < \varepsilon$$

$\forall z, w \in K$, $|z-w| < \delta$

$\Leftarrow$ Given $\{f_n\} \subset \mathcal{C}(\Omega; \hat{\mathbb{C}})$,

we simply prove that it admits a (uniform) convergent subsequence on any compact subset $K \subset \Omega$

It suffices to show that there is a Cauchy subsequence (in $\max_K d_{\hat{\mathbb{C}}}$ norm)

Let $Q = \{z \in K \mid \text{rational real and imaginary part}\}$

By diagonal argument, $\exists$ subsequence of $\{f_n\}$ (still denote it by $\{f_n\}$) such that $\{f_n(z)\}$ is Cauchy in $\hat{\mathbb{C}}$ $\forall z \in Q$

Given $\varepsilon > 0$, $\exists\, \delta$ such that $d_{\hat{\mathbb{C}}}(f_n(z), f_n(w)) < \frac{\varepsilon}{3}$ $\quad \forall\, n \geq 1,\ z, w \in K,\ |z-w| < \delta$

$\{B(z;\delta)\}_{z \in Q}$ is an open cover for K

$\Rightarrow \exists$ finite subcover $\{B(z_j;\delta)\}_{j=1}^{J}$

$\Rightarrow \exists\, N$ such that $d_{\hat{\mathbb{C}}}(f_n(z_j), f_m(z_j)) < \frac{\varepsilon}{3}$ $\quad \forall\, 1 \leq j \leq J,\ n, m \geq N$

Then, for any $w \in K$

$\exists\, z_j$ such that $|w - z_j| < \delta$

$$\Rightarrow d_{\hat{\mathbb{C}}}(f_n(w), f_m(w)) \leq d_{\hat{\mathbb{C}}}(f_n(w), f_n(z_j)) + d_{\hat{\mathbb{C}}}(f_n(z_j), f_m(z_j)) + d_{\hat{\mathbb{C}}}(f_m(z_j), f_m(w))$$

$$< \varepsilon \qquad \forall\, n, m \geq N$$

※

lemma $\{f_n\}$: meromorphic (analytic) on Ω if $f_n \to f$ in the ρ-metric (or equivalently, uniformly on compact sets)

Then, f is either meromorphic (analytic) or identically ∞

pf: • if $f(z_0) \neq \infty$, f is analytic near z_0 (due to Weierstrass)

if $f(z_0) = \infty$, $\frac{1}{f_n} \to \frac{1}{f}$ uniformly on a neighborhood of z_0 in $d_{\hat{\mathbb{C}}}$-metric $\Longrightarrow$ in Euclidean metric

again, the Weierstrass theorem applies away from ∞

• if f_n is analytic and $f(z_0) = \infty$

$\frac{1}{f_n}$: nowhere zero on $B(z_0;\varepsilon)$ $\xrightarrow[\text{on } B(z_0;\varepsilon)]{\text{uniformly}}$ $\frac{1}{f}$ vanishes at z_0

$\Rightarrow \frac{1}{f} \equiv 0 \quad \Rightarrow f \equiv \infty$

※

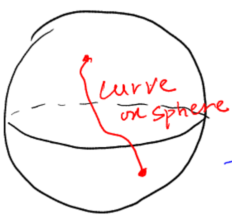
thm (Marty) $\mathcal{F}$: meromorphic functions on Ω

$\mathcal{F}$ is normal if and only if

$\rho(f) = \dfrac{2|f'(z)|}{1+|f(z)|^2}$ is locally bounded (i.e. on any compact set K $\exists\, M$ so that $\rho(f) \leq M \quad \forall f \in \mathcal{F},\ z \in K$)

pf: $\Leftarrow$) equi-continuity on compact sets?

$$z(t) \longrightarrow \left(\frac{2z(t)}{1+|z(t)|^2}, \frac{1-|z(t)|^2}{1+|z(t)|^2} = \frac{2}{1+|z(t)|^2} - 1\right) \in \mathbb{C} \times \mathbb{R}$$



$$\text{arc-length}: \frac{d}{dt} \int \left(\frac{2z'}{1+|z|^2} - \frac{2z(\bar{z}z' + z\bar{z}')}{(1+|z|^2)^2},\ -\frac{2(\bar{z}z' + z\bar{z}')}{(1+|z|^2)^2}\right) \qquad z' = \frac{dz(t)}{dt}$$

$$= \frac{2}{(1+|z|^2)^2}(z' - z^2\bar{z}')$$

$$|\ -\ |^2 = \frac{4}{(1+|z|^2)^4}\left(|z' - z^2\bar{z}'|^2 + (\bar{z}z' + z\bar{z}')^2\right) = \frac{4}{(1+|z|^2)^2}|z'|^2$$

$$\text{arc-length} = \int \frac{2|z'|}{1+|z|^2}\, dt$$

Given $p \in \Omega$, $\exists\, \varepsilon > 0$ such that $\overline{B(p,\varepsilon)} \subset \Omega$

For any $z, w \in \overline{B(p,\varepsilon)}$, $d_{\hat{\mathbb{C}}}(f(z), f(w)) \leq \displaystyle\int_\gamma \frac{2|f'|}{1+|f|^2}|dz|$

By the locally bounded of $\rho(f)$ for $f \in \mathcal{F}_1$

$$d_{\hat{\mathbb{C}}}(f(z), f(w)) \le M|z-w| \quad \forall z, w \in \overline{B(p;\varepsilon)}$$

Thus, $\mathcal{F}_1$ is equi-continuous on $\overline{B(p;\varepsilon)}$ $\Rightarrow$ normal

$\Rightarrow$)

- if z is a pole of f, $\frac{1}{f}$ is analytic near f

$$\rho(\tfrac{1}{f}) = \frac{2|f'/f^2|}{1+|1/f|^2} = \frac{2|f'|}{1+|f|^2} = \rho(f)$$

- Given $p \in \Omega$, choose a compact neighborhood K of p

  By assumption and Arzela's theorem, $\mathcal{F}_1$ is equi-continuous on K.

  For $\varepsilon = \frac{1}{1000}$, $\exists\, \delta > 0$ such that $d_{\hat{\mathbb{C}}}(f(z), f(w)) < \frac{1}{1000}$ for $z, w \in K$, $|z-w| < 4\delta$, any $f \in \mathcal{F}_1$

  Consider $B(p;\delta)$

  <u>claim</u> $\rho(f)$ is uniform bounded on $B(p;\delta)$

  case 1 : $|f(p)| \le 1$ case 2 : $|f(p)| > 1$ (including p is a pole)

- case 1

  $|f(w)| \le 2 \quad \forall w \in \overline{B(p;2\delta)}$

  For any $z \in B(p;\delta)$,

$$f'(z) = \frac{1}{2\pi i} \int_{|w-p|=2\delta} \frac{f(w)}{w-z}\, dw$$

$$|f'(z)| \le \frac{1}{2\pi} \frac{2}{\delta^2} 4\pi\delta = \frac{4}{\delta}$$

$$\Rightarrow \rho(f) = \frac{2|f'(z)|}{1+|f|^2} \le 2|f'(z)| \le \frac{8}{\delta}$$

- case 2

  $|f(w)| \ge \frac{1}{2} \quad \forall w \in \overline{B(p;2\delta)}$

  Consider $g(w) = 1/f(w)$ $\Rightarrow |g(w)| \le 2 \quad \forall w \in \overline{B(p;2\delta)}$

  Similarly, $\underset{\displaystyle \rho(f)}{\overset{\rho(g)}{\shortparallel}} \le \frac{8}{\delta} \quad \forall z \in B(p;\delta)$

Hence, $\rho(f)$ is uniformly bounded on $B(p;\delta)$ #