# IX. prime number theorem [ Lang, complex analysis, ch. XVI ]

## §1 growth of prime numbers

rough picture

behavior of $\pi(x)$ or $\varphi(x)$ near infinity $\leftrightsquigarrow$ singular behavior of the Lapace transform at $u=0$

$\updownarrow$ related

zeta function

$\pi(x) = \#\{\text{primes} \le x\}$

thm 1 $\pi(x) \sim \frac{x}{\log x}$ namely. $\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$

$\varphi(x) = \sum_{p\le x} \log p$ ( roughly $\log x \, \pi(x) \gtrsim x$ )

lemma 2 $\int_1^\infty \frac{\varphi(x) - x}{x^2} dx < \infty$ $\longrightarrow$ diverges $\Rightarrow \varphi(x) = x$ + smaller terms

prop 3 $\varphi(x) \sim x$

Let us postpone the proof of the lemma 2.

pf for prop 3 : $\underline{\lim}_{x\to\infty} \frac{\varphi(x)}{x} \gtrsim 1$

Given $0<\lambda<1$, $\underline{\lim}_{x\to\infty} \frac{\varphi(x)}{x} \ge \lambda$ ? Is $\{x \mid \varphi(x) \le \lambda x\}$ bounded ?

Suppose the set is unbounded $\Rightarrow \exists \{x_j\} \subset \{x \mid \varphi(x) \le \lambda x\}$ with $\lambda x_{j+1} \ge x_j$

$$\int_{\lambda x_j}^{x_j} \frac{\varphi(t)-t}{t^2} dt \le \int_{\lambda x_j}^{x_j} \frac{\varphi(x_j)-t}{t^2} dt \le \int_{\lambda x_j}^{x_j} \frac{\lambda x_j - t}{t^2} dt \qquad t = x_j s$$

$$\le \int_\lambda^1 \frac{\lambda - s}{s^2} ds < 0 \quad \Rightarrow \sum_j \int_{\lambda x_j}^{x_j} \frac{\varphi(t)-t}{t^2} dt \text{ diverges}$$

Similarly $\overline{\lim}_{x\to\infty} \frac{\varphi(x)}{x} \le 1$ ※

pf for thm 1 :

$\varphi(x) = \sum_{p\le x} \log p \le \sum_{p\le x} \log x = \pi(x) \log x \quad \Rightarrow \underline{\lim}_{x\to\infty} \frac{\pi(x)}{x/\log x} \ge 1$

$\varphi(x) = \underbrace{\sum_{p\le x} \log p \ge \sum_{x^{1-\varepsilon}\le p\le x}}_{\text{most primes}} \log p \ge \sum_{x^{1-\varepsilon}\le p\le x} (1-\varepsilon) \log x$

$= (1-\varepsilon) \log x \, (\pi(x) - \pi(x^{1-\varepsilon}))$

$\Rightarrow \frac{\pi(x)}{\varphi(x)/\log x} \le \frac{1}{1-\varepsilon} + \frac{\pi(x^{1-\varepsilon})}{\varphi(x)/\log x}$ but $\pi(x^{1-\varepsilon}) \le x^{1-\varepsilon}$ and $\varphi(x) \sim x$

$\Rightarrow \overline{\lim}_{x\to\infty} \frac{\pi(x)}{x/\log x} \le \frac{1}{1-\varepsilon}$ for any $\varepsilon > 0$ ※

## §2 relation between $\varphi(x)$ and $\zeta(s)$

define $\Phi(s) = \sum_{\text{prime}} \frac{\log p}{p^s}$ : analytic for $\mathrm{Re}\, s > 1$

[ subseries of $\sum_n \frac{\log n}{n^s}$ converges for $\mathrm{Re}\, s > 1$ ]

idea/goal
$$\begin{cases} \Phi(s) : \text{crucial part of } -\frac{\zeta'(s)}{\zeta(s)} \\ \Phi(s) = s \int_1^\infty \frac{\varphi(x)}{x^{s+1}} dx \end{cases}$$

**thm 4** $\Phi(s)$ extends to a meromorphic function for $\operatorname{Re} s > \frac{1}{2}$

Furthermore, $s=1$ is the only pole for $\operatorname{Re} s \geq 1$, which is of order 1 and has residue 1

pf: $$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \operatorname{Re} s > 1$$

$$\Rightarrow -\frac{\zeta'}{\zeta} = \sum_p \frac{\log p}{p^s - 1} \qquad \frac{1}{p^s-1} = \frac{1}{p^s}\left(\frac{1}{1-p^{-s}}\right) = \frac{1}{p^s}\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$

$$= \frac{1}{p^s} + (\text{smaller})$$

$$\Rightarrow -\frac{\zeta'}{\zeta} = \Phi(s) + \sum_p h_p(s) \qquad \left|\frac{1}{p^s-1} - \frac{1}{p^s}\right| = \left|\frac{1}{p^s(p^s-1)}\right| \leq \frac{100}{|p^{2s}|} \text{ for } \operatorname{Re} s > \frac{1}{2}$$

($s$ away from 0)

where $|h_p(s)| \leq 100 \dfrac{\log p}{|p^{2s}|}$ for $\operatorname{Re} s > \frac{1}{2}$

$\Rightarrow \sum_p h_p(s)$ : analytic on $\operatorname{Re} s > \frac{1}{2}$

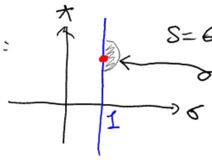
Hence, we extend $\Phi(s)$ to $\boxed{\operatorname{Re} s > \frac{1}{2}}$

poles of zeros comes from the zeros and poles of $\zeta(s)$

$\boxed{\text{claim}}$ $\zeta(s) \neq 0$ when $\operatorname{Re} s = 1$ #

**prop** $\zeta(s)$ has no zero on $\operatorname{Re} s = 1$

pf:




$s = \sigma + it$ suppose not, examine the local behavior

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \exp\left(\sum_p -\log\left(1 - \frac{1}{p^s}\right)\right)$$

$$= \exp\left(\sum_p \sum_{n=1}^{\infty} \frac{1}{n(p^s)^n}\right)$$

$$|\zeta(s)| = \exp \sum_p \sum_{m=1}^{\infty} \frac{\cos(nt \log p)}{n p^{n\sigma}}$$

$$\boxed{\begin{aligned} & s = \sigma + it \\ & p^{ns} = p^{n\sigma} e^{int \log p} \\ & (p^{ns})^{-1} = (p^{n\sigma})^{-1} e^{-int \log p} \end{aligned}}$$

$$\Rightarrow \zeta^3(\sigma)\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)| = \exp \sum_p \sum_{m=1}^{\infty} \frac{3 + 4\cos(nt \log p) + \cos(2nt \log p)}{n p^{n\sigma}}$$

But $3 + 4\cos\theta + \cos(2\theta) = 2(1+\cos\theta)^2 \geq 0$

$$\Rightarrow \zeta^3(\sigma)\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)| \geq 1 \quad \text{for any } \sigma > 1$$

If $\zeta(1+it) = 0$ for some $t \neq 0$,

consider $\zeta(\sigma)$, $\zeta(\sigma+it)$, $\zeta(\sigma+2it)$ as $\sigma \to 1$

$\downarrow \frac{1}{\sigma-1}$ $\quad\downarrow$ zero $\quad\downarrow$ bounded $\quad$ goes to 0 as $\sigma \to 1$ $\rightarrow\leftarrow$

**prop 5** $\Phi(s) = s \displaystyle\int_1^{\infty} \frac{\varphi(x)}{x^{s+1}}\, dx$ for $\operatorname{Re} s > 1$

pf: $$\int_1^{\infty} \frac{\varphi(x)}{x^{s+1}}\, dx = \left(\int_1^{p_1} + \int_{p_1}^{p_2} + \cdots \frac{\varphi(x)}{x^{s+1}}\, dx\right)$$

$$s\int_{p_n}^{p_{n+1}} \frac{\varphi(x)}{x^{s+1}}\, dx = s\int_{p_n}^{p_{n+1}} \frac{\sum_{k=1}^{n} \log p_k}{x^{s+1}}\, dx = \left(\sum_{k=1}^{n} \log p_k\right)\left(\frac{1}{p_n^s} - \frac{1}{p_{n+1}^s}\right)$$

$$\Rightarrow s\int_1^{\infty} \frac{\varphi(x)}{x^{s+1}}\, dx = \sum_{n=1}^{\infty} \log p_n \left(\left(\frac{1}{p_n^s} - \frac{1}{p_{n+1}^s}\right) + \left(\frac{1}{p_{n+1}^s} - \frac{1}{p_{n+2}^s}\right) + \cdots\right) = \sum_p \frac{\log p}{p^s} = \Phi(s)$$

#

## §3 proof of lemma 2 $\left( \int_1^\infty \frac{\varphi(x)-x}{x^2}\,dx \text{ converges} \right)$

$$\Phi(s) = s\int_1^\infty \frac{\varphi(x)}{x^{s+1}}\,dx \qquad \mathrm{Re}\, s > 1$$

$$\frac{1}{s-1} = \int_1^\infty \frac{x}{x^{s+1}}\,dx \qquad \text{also for } \mathrm{Re}\, s > 1$$

$$\underbrace{\frac{\Phi(s)}{s} - \frac{1}{s-1}}_{\text{bounded as } s\to 1} = \int_1^\infty \frac{\varphi(x)-x}{x^{s+1}}\,dx$$

bounded as $s \to 1$ R.H.S ?

two ingredients
- thm 6 (Chebyshev) $\varphi(x) = O(x)$ i.e. $|\varphi(x)| \le C|x|$ for $|x| \ge 1$
  [skip the proof]
- Laplace transform $g(u) = \int_0^\infty f(t)\, e^{-ut}\,dt$ $\mathrm{Re}\, u > 0$.

  $f$: bounded, piecewise continuous

  lemma 7 $\Rightarrow$ If $g(u)$ extends to analytic function for $\mathrm{Re}\, u \ge 0$

  Then, $\int_0^\infty f(t)\,dt$ exists, and is $g(0)$

proof of lemma 2:

$s = u+1$
$$\frac{\Phi(u+1)}{u+1} - \frac{1}{u} = \int_1^\infty \frac{\varphi(x)-x}{x^{u+2}}\,dx \qquad x = e^t$$
$$= \int_0^\infty \frac{\varphi(e^t)-e^t}{e^t}\, e^{-ut}\,dt$$

bounded by Chebyshev

thm 4 $\Rightarrow$ extends to an analytic function for $\mathrm{Re}\, u = \mathrm{Re}\, s + 1 \ge 0$

lemma 7 $\Rightarrow$ the integral converges for $u=0$ $(s=1)$ ※

rmk large behavior of $f(t)$ $\xrightarrow{\text{Fourier}}$ local behavior of $g(u)$ at $\mathrm{Re}\, u = 0$

proof of lemma 7:

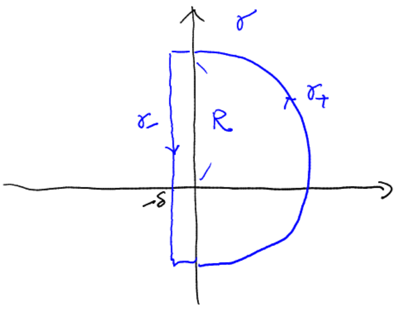
Consider $g_T(u) = \int_0^T f(t)\, e^{-ut}\,dt$ : entire

$g_T(0) = \int_0^T f(t)\,dt$

We have to show that $\lim_{T\to\infty} g_T(0) = g(0)$

When $\mathrm{Re}(z) > 0$ $\quad g(u) - g_T(u) = \int_T^\infty f(t)\, e^{-ut}\,dt \Rightarrow |g(u) - g_T(u)| \le \frac{C}{\mathrm{Re}\, u}\, e^{-(\mathrm{Re}\, u)T}$

We can estimate $g(0) - g_T(0)$ by $g(u) - g_T(u)$ with the help of the Cauchy integral formula




$$g(0) - g_T(0) = \frac{1}{2\pi i}\int_\gamma (g(u) - g_T(u))\, h_T(u)\, \frac{du}{u}$$

$h_T(u)$: analytic with $k(0) = 1$

$$= \frac{1}{2\pi i}\int_{\gamma_-} + \frac{1}{2\pi i}\int_{\gamma_+}$$

$\int_{\gamma_-} \to 0$ as $T\to\infty$; $\int_{\gamma_+}$ small as $R \gg 1$

goal

1° Choose $h_T(u) = e^{Tu}\left(1 + \frac{u^2}{R^2}\right) = e^{Tu}\,\frac{u}{R}\left(\frac{R}{u} + \frac{u}{R}\right) \Rightarrow |h(u)| \le e^{T\,\mathrm{Re}\, u}\,\frac{2\,\mathrm{Re}\, u}{R}$ for $u \in \gamma_+$

$e^{Tu}$: use it to help: transform $T$ from $g_T$ to $g$

$$\Rightarrow \left| \frac{1}{2\pi i} \int_{\gamma_+} \left( g(u) - g_T(u) \right) h_T(u) \frac{du}{u} \right| \leq \frac{C'}{R}$$

T-independent estimate

$2^\circ$ For $\displaystyle\int_{\gamma_-} g_T(u)\, h_T(u) \frac{du}{u}$ : $\quad |g_T(u)| \leq C \int_0^T e^{-(\mathrm{Re}\,u)t}\, dt \leq \dfrac{e^{-T\,\mathrm{Re}\,u}}{-\mathrm{Re}\,u}$

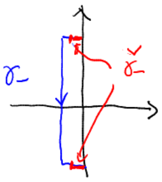
$$\left| \int_{\gamma_-} g_T(u)\, h_T(u) \frac{du}{u} \right| = \left| \int_{-\gamma_+} g_T(u)\, h_T(u) \frac{du}{u} \right| \leq \frac{C''}{R}$$

$3^\circ$ For $\displaystyle\int_{\gamma_-} g(u)\, h_T(u) \frac{du}{u}$ : $\quad g(u)\, e^{Tu} \left(1 + \frac{u^2}{R^2}\right)$ T-independent analytic function

$|e^{Tu}| \leq 1$ for $u \in \gamma_-$

Sum up: given $\varepsilon > 0$. choose $R \gg 1$

so that $\dfrac{C'}{R}$ and $\dfrac{C''}{R} \leq \dfrac{\varepsilon}{4}$




choose $\delta$ so that $g(u)$ is analytic on the region enclosed by $\gamma$

$\Rightarrow \exists\ \check{\gamma}_-$ : short enough so that $\left| \displaystyle\int_{\check{\gamma}_-} g(u)\, h_T(u) \frac{du}{u} \right| \leq \dfrac{\varepsilon}{4}$

$\gamma_- = \hat{\gamma}_+ \cup \hat{\gamma}_-$ . $\exists\ T \gg 1$ so that $\left| \displaystyle\int_{\check{\gamma}_+} g(u)\, h_T(u) \frac{du}{u} \right| \leq \dfrac{\varepsilon}{4}$

use $e^{uT}$

※