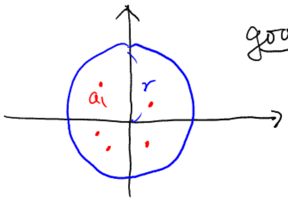
VII. entire function : Jensen's formula and Hadamard factorization

<u>recall</u> (Weierstrass) $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1-\frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}\right)$

genus $= \max \left\{ \deg g(z) \;,\; \min_{h \in \mathbb{Z}_{\geq 0}} \sum_n \left(\frac{1}{|a_n|}\right)^{h+1} < \infty \right\}$



<u>goal</u> genus / growth of zeros of $f(z)$

$\quad \leftrightsquigarrow$ growth rate of $\sup_{|z|=r} |f(z)|$ in r

§1 Jensen's formula

$f(z)$: entire $\quad \{a_1, \cdots, a_n\}$ = zeros in $B(0;r)$ $\quad \leftarrow$ repeat if multiplicity > 1

Jensen $\updownarrow$ relates

$|f(z)|$ for $|z| = r$

1° basics of harmonic function

- $f = u + iv$: analytic $\Rightarrow$ u & v are harmonic, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}} = 0$

  If f is nowhere zero $\Rightarrow$ $\log |f|$ is harmonic
  $= \frac{1}{2} \log |f|^2$

  recall
  $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$
  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$

  $\left[\begin{array}{l} \frac{\partial}{\partial \bar{z}} \log |f|^2 = (f\bar{f})^{-1} f \frac{\partial \bar{f}}{\partial \bar{z}} = \bar{f}^{-1} \frac{\partial \bar{f}}{\partial \bar{z}} \\ \frac{\partial^2}{\partial z \partial \bar{z}} \log |f|^2 = \frac{\partial}{\partial z}\left(\bar{f}^{-1} \frac{\partial \bar{f}}{\partial \bar{z}}\right) = 0 \qquad \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial \bar{f}}{\partial z} = 0 \end{array} \right]$

- u = harmonic on $\mathbb{R}^2$ (or some ball at 0) $\Rightarrow$ $u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{i\theta})\, d\theta$ $\leftarrow$ polar coordinate

  (the mean value property)

  [The conjugate harmonic v can be constructed by $v_y = u_x$, $v_x = -u_y$

  $\Rightarrow$ $u + iv$: entire, $(u+iv)(0) = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{(u+iv)(z)}{z}\, dz$ by Cauchy

  $\Rightarrow$ $u(0) = \mathrm{Re}\left\{ \frac{1}{2\pi} \int_0^{2\pi} (u+iv)(e^{i\theta})\, d\theta \right\} = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})\, d\theta$

  You can also prove it by the power series of z ※]

- Hence, if $f(z)$: entire without zeros on $\overline{B(0;\rho)}$

  $\Rightarrow$ $\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\theta})|\, d\theta$

  <u>rmk</u> the formula holds as well when f has zeros when $|z| = \rho$

  [<u>key</u> $f(z) = 1 - z$ on $B(0;1)$ $\Rightarrow$ $f(0) = 1$ $\quad \log |f(0)| = 0$

  Also, $\int_0^{2\pi} \log |1 - e^{i\theta}|\, d\theta = \frac{1}{2} \int_0^{2\pi} \log(2 - 2\cos\theta)\, d\theta \overset{\boxed{\text{check}}}{=} 0$ ※]

<u>Q</u> what happens if $f(z)$ has zero in $B(0;\rho)$?

2° Jensen's formula. $f(z)$: entire (non-trivial)

Let $\{a_1, \cdots, a_n\}$ be the zeros of $f(z)$ within $|z|<1$ (multiple zeros being repeated)

Then $f(z) \Big/ \prod_{k=1}^{n}(z-a_n)$ has no zeros for $|z|<1$

But the value changes on the boundary, $|z|=1$

idea / recall $z \longmapsto \frac{z-a}{1-\bar{a}z}$ is an automorphism of the unit disk and sends boundary to boundary

$\Rightarrow$ Consider $F(z) = f(z)\prod_{k=1}^{n}\frac{1-\bar{a}_k z}{z-a_k}$ $\begin{cases} F(z) \text{ has no zeros within } |z|<1 \\ |F(z)| = |f(z)| \text{ for } |z|=1 \end{cases}$

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|\,d\theta$$
$$= \log|f(0)| + \sum_{k=1}^{n}\log\frac{1}{|a_k|}$$

For $|z| \le \rho$,

$$\log|f(0)| + \sum_{k=1}^{n}\log\frac{\rho}{|a_k|} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{i\theta})|\,d\theta \quad \left(\text{Jensen's formula}\right)$$

relation between zeros within $|z|<\rho$ and $|f(z)|$ for $|z|=\rho$

3° Again on $\overline{B(0;1)}$, instead of $f(0)$, any $z_0 \in B(0;1)$ gives a balancing relation.

idea sends $0 \mapsto z_0$ — disk automorphism, inverse $= \frac{w-z_0}{1-\bar{z}_0 w}$

Consider $g(z) = f\Big(\underbrace{\frac{z+z_0}{1+\bar{z}_0 z}}_{=w}\Big)$, $g(0) = f(z_0)$

zeros of g : $\frac{a_j - z_0}{1-\bar{z}_0 a_j}$

The Jensen's formula for g reads

$$\log|f(z_0)| + \sum_{j=1}^{n}\log\left|\frac{1-\bar{z}_0 a_j}{a_j - z_0}\right| = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\frac{e^{i\theta}+z_0}{1+\bar{z}_0 e^{i\theta}}\right)\right|\,d\theta$$

Let $\frac{e^{i\theta}+z_0}{1+\bar{z}_0 e^{i\theta}} = e^{i\phi} \Rightarrow e^{i\theta} = \frac{e^{i\phi}-z_0}{1-\bar{z}_0 e^{i\phi}}$

$$i\,d\theta = e^{-i\theta}\,de^{i\theta} = \frac{1-\bar{z}_0 e^{i\phi}}{e^{i\phi}-z_0}\,\frac{(1-\bar{z}_0 e^{i\phi})e^{i\phi} + (e^{i\phi}-z_0)\bar{z}_0 e^{i\phi}}{(1-\bar{z}_0 e^{i\phi})^2}\,i\,d\phi$$

$$d\theta = \frac{1-|z_0|^2}{(e^{i\phi}-z_0)(e^{-i\phi}-\bar{z}_0)}\,d\phi = \operatorname{Re}\frac{e^{i\phi}+z_0}{e^{i\phi}-z_0}\,d\phi$$

$$\Rightarrow \log|f(z_0)| + \sum_{j=1}^{n}\log\left|\frac{1-\bar{z}_0 a_j}{a_j-z_0}\right| = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\frac{e^{i\phi}+z_0}{e^{i\phi}-z_0}\log|f(e^{i\phi})|\,d\phi$$

Similarly, for $|z| \le \rho$

$$\log|f(z_0)| + \sum_{j=1}^{n}\log\left|\frac{\rho^2-\bar{z}_0 a_j}{\rho(a_j-z_0)}\right| = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\left(\frac{\rho e^{i\phi}+z_0}{\rho e^{i\phi}-z_0}\right)\log|f(\rho e^{i\phi})|\,d\phi$$

(Poisson-Jensen formula)

§2 Hadamard's theorem

Suppose that $f(z)$ is an entire function of finite genus $= h$ (with $f(0) \neq 0$)

$$\Rightarrow \quad f(z) = e^{\overset{\text{a degree} \le h \text{ polynomial}}{g(z)}} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h\right)$$

<u>Q</u> relation between $\sup_{|z|=r} \log|f(z)|$ and the genus h ?

<u>discussion</u>

$$\log|f(z)| = \underbrace{\mathrm{Re}(g(z))}_{\sim\, |b_h z^h| \text{ as } |z| \to \infty} + \underbrace{\sum_{n=1}^{\infty} \log\left|\left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h\right)\right|}_{\le \left(\frac{2}{h+1} \sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}}\right) r^{h+1} \text{ for } |z| < r}$$

intuitively, $|f(z)| \le e^{|z|^h} \cdot e^{|z|^{h+1}}$ $\quad (\Rightarrow \lambda \le h+1)$

<u>defn</u> $M(r) = \sup_{|z|=r} |f(z)|$

<u>order</u> of $f(z)$ is defined by $\displaystyle \lambda = \overline{\lim_{r \to \infty}} \frac{\log \log M(r)}{\log r}$

<u>thm</u> (Hadamard factorization) the genus and order of an entire function satisfies $h \le \lambda \le h+1$ (note that $h \in \mathbb{N}_{\ge 0} \cup \{\infty\}$)

pf: (For $\lambda \le h+1$) Suppose that $f(z)$ has finite genus h

$$\boxed{\begin{gathered} \sum_{n=1}^{\infty} \left(\frac{1}{|a_n|}\right)^{h+1} < \infty \\ \deg g(z) \le h \end{gathered}}$$

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h\right)$$

$$\log|f(z)| = \mathrm{Re}\, g(z) + \sum_{n=1} \log\left|E_h\left(\frac{z}{a_n}\right)\right|$$

[where $E_h(u) = (1-u) \exp\left(u + \frac{1}{2}u^2 + \cdots + \frac{1}{h}u^h\right)$]

$$\le 2 \max\left\{ |g(z)|,\ \sum_{n=1}^{\infty} \left| \log\left(1 - \frac{z}{a_n}\right) + \frac{z}{a_n} + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h \right| \right\}$$

clearly, $e^{g(z)}$ has order $\le h$,

and it remains to examine the canonical product.

<u>key issue</u> $\log|E_h(u)| \le |\log E_h(u)| < \frac{1}{h+1} \frac{|u|^{h+1}}{1-|u|}$ only for $|u| < 1$

need an estimate for $|u| > 1$

$$\boxed{\begin{gathered} \text{goal} \\ \log|E_h(u)| < C_h |u|^{h+1} \end{gathered}}$$

$$\boxed{\begin{gathered} \text{given any } z \\ \left|\frac{z}{a_n}\right| < 1 \text{ except for finitely many } n \\ \rightsquigarrow \text{ need a good estimate} \end{gathered}}$$

<u>observation</u> $\begin{cases} \log|E_h(1)| = -\infty \\ \log|E_h(u)| \sim |u|^h & \text{for } |u| \gg 1 \end{cases}$

$h = 0$: $\log|1-u| \le \log(1+|u|) \le |u| \quad \forall\, |u|$

$$\log|E_h(u)| \le \log|E_{h-1}(u)| + |u|^h$$

induction $\Rightarrow$ $\log|E_h(u)| \le (h+1)|u|^h \le (h+1)|u|^{h+1}$ <u>when $|u| \ge 1$</u>

<u>when $|u| < 1$</u> $(|u|^h > |u|^{h+1})$, $\log|E_h(u)| = (1-|u|)\log|E_h(u)| + |u| \log|E_h(u)|$

$$\le |u|^{h+1} + |u| \log|E_{h-1}(u)| + |u|^{h+1}$$

induction $\Rightarrow$ $\log|E_h(u)| \le (2h+1)\,|u|^{h+1}$ when $|u|<1$

Thus. $\log|E_h(u)| \le (2h+1)\,|u|^{h+1} \quad \forall u$

Then, $\sum_{n=1}^{\infty} \log\left|E_h\left(\frac{z}{a_n}\right)\right| \le (2h+1)\left(\sum_{n=1}^{\infty}\frac{1}{|a_n|^{h+1}}\right)|z|^{h+1} \Rightarrow \lambda \le h+1$

(For $\lambda \ge h$) Suppose that $f(z)$ has finite order λ

Let $h = \max\{m\in\mathbb{Z} \mid m \le \lambda\}$ $\quad (\Rightarrow h+1>\lambda)$

i) $\sum_{n=1}^{\infty}\frac{1}{|a_n|^{h+1}} < \infty$?

Let $\nu(\rho)$ be the number of zeros within $\overline{B(0;\rho)}$

$\Rightarrow \nu(\rho) \le \left(\log M(2\rho) - \log|f(0)|\right)/\log 2$ (Jensen and HW)

$\Rightarrow \lim_{\rho\to 0}\frac{\nu(\rho)}{\rho^{\lambda+\varepsilon}} = 0$ for any $\varepsilon>0$ (due to order λ)

Assume $0<|a_1|\le|a_2|\le\cdots \Rightarrow n \le \nu(|a_n|) < |a_n|^{\lambda+\varepsilon}$ for $n\gg 1$

Choose $\varepsilon>0$ so that $h+1>\lambda+\varepsilon$

$\Rightarrow \frac{1}{|a_n|^{h+1}} = \left(\frac{1}{|a_n|^{\lambda+\varepsilon}}\right)^{\frac{h+1}{\lambda+\varepsilon}} = \left(\frac{1}{n}\right)^{\frac{h+1}{\lambda+\varepsilon}>1} \Rightarrow$ converges

ii) $g(z)$: a polynomial $\le h$? $\Leftrightarrow g^{(h+1)}\equiv 0$

$f(z) = e^{g(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n}+\cdots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h\right)$

$\frac{f'}{f} = g' + \sum_{n=1}^{\infty}\left((z-a_n)^{-1} + \frac{1}{a_n}\left(1+\frac{z}{a_n}+\cdots+\left(\frac{z}{a_n}\right)^{h-1}\right)\right)$ $\boxed{\text{relate } g' \text{ to } M(\rho) \text{ ?}}$

$\log|f(z)| = -\sum_{k=1}^{n}\log\left|\frac{\rho^2-\bar{a}_k z}{\rho(z-a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\,|f(\rho e^{i\theta})|\,d\theta$

$2\frac{\partial}{\partial z}$

$\log|f(z)| = \frac{1}{2}\log f\bar{f} \qquad 2\frac{\partial}{\partial z}\log|f(z)| = \frac{f'\bar{f}}{f\bar{f}} = \frac{f'}{f}$

$\frac{f'}{f} = \sum_{k=1}^{\nu(\rho)}(z-a_k)^{-1} + \sum_{k=1}^{\nu(\rho)}\bar{a}_k(\rho^2-\bar{a}_k z)^{-1} + \frac{1}{2\pi}\int_0^{2\pi}\frac{2\rho e^{i\theta}}{(\rho e^{i\theta}-z)^2}\log|f(\rho e^{i\theta})|\,d\theta$

$\left(\frac{f'}{f}\right)^{(h)} = -h!\sum_{k=1}^{\nu(\rho)}(a_k-z)^{-(h+1)} + h!\sum_{k=1}^{\nu(\rho)}\bar{a}_k^{\,h+1}(\rho^2-\bar{a}_k z)^{-(h+1)}$

$+ \frac{(h+1)!}{2\pi}\int_0^{2\pi}\frac{2\rho e^{i\theta}}{(\rho e^{i\theta}-z)^{h+2}}\log|f(\rho e^{i\theta})|\,d\theta$

Given z, consider $\rho>2|z|$ going to ∞

$\left|\frac{(h+1)!}{2\pi}\int_0^{2\pi}\frac{2\rho e^{i\theta}}{(\rho e^{i\theta}-z)^{h+2}}\log|f(\rho e^{i\theta})|\,d\theta\right|$

$= \left|\frac{(h+1)!}{2\pi}\int_0^{2\pi}\frac{2\rho e^{i\theta}}{(\rho e^{i\theta}-z)^{h+2}}\left(\log|f(\rho e^{i\theta})| - \log M(\rho)\right)d\theta\right|$

$\left[\begin{array}{l}|\rho e^{i\theta}-z| \\ \ge \rho-|z| > \frac{\rho}{2}\end{array}\right]$ $\le \frac{(h+1)!}{2\pi}\,2^{h+3}\rho^{-(h+1)}\int_0^{2\pi}\underbrace{\left(\log M(\rho)-\log|f(\rho e^{i\theta})|\right)}_{\ge 0}d\theta$

But $\underbrace{\rho^{-(h+2)}\log M(\rho)\to 0}_{\text{(order)}}$ and $\underbrace{\frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{i\theta})|\,d\theta \ge \log|f(0)|}_{\text{(Jensen)}}$

$\Rightarrow$ the integral term $\to 0$ as $\rho\to\infty$

The second term : $\sum_{k=0}^{\nu(\rho)} \overline{a_k}^{\,h+1} (\rho^2 - \overline{a_k} z)^{-(h+1)} \leq \left(\frac{z}{\rho}\right)^{h+1} n(\rho) \xrightarrow[\text{see i)}]{} 0$ as $\rho \to \infty$

$\overline{a_k}^{\,h+1} \leq \rho^{h+1}$, $\quad (\rho^2 - \overline{a_k} z)^{-(h+1)} \leq \left(\frac{\rho^2}{2}\right)^{-(h+1)}$

$\Rightarrow \left(\frac{f'}{f}\right)^{(h)} = -h! \sum_{k=0}^{\infty} (a_n - z)^{-(h+1)}$ compare with the Weierstrass product

$\Rightarrow g^{(n+1)} \equiv 0$ $\divideontimes$

<u>Cor</u> an entire function with finite order $\lambda \notin \mathbb{Z}$ has infinitely many zeros / assumes each value an infinite number of times

Pf: If f has only finitely many zeros,

$f(z) = e^{g(z)} (z-a_1) \cdots (z-a_n)$

By Hadamard, $g(z)$ is a polynomial of order $\leq \lambda$.

But it is not hard to see that f & e^g have the same order. Hence, order $(f) = \deg(g) \in \mathbb{Z}_{\geq 0}$ $\rightarrow\leftarrow$