infinite sum, infinite product

elementary transcendental functions & other special functions

VI. sums, products and Gamma function

§1. partial fraction of meromorphic functions [A, §2.1 of ch. 5]

recall $f(z)$: rational function (on $\mathbb{C}$)

$$f(z) = \frac{p(z)}{q(z)} = \sum_{k=1}^{n} P_k\left(\frac{1}{z-b_k}\right) + g(z)$$

b_k: zeros of $q(z)$ $\qquad$ $g(z)$: polynomial

P_k: polynomials without constant term

Q Can we develop similar representation for $f(z)$: meromorphic on $\mathbb{C}$?

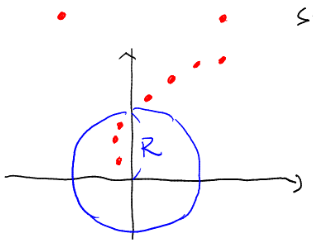
b_k: poles of $f(z)$ $\qquad$ $g(z)$: entire (analytic on $\mathbb{C}$)

P_k: singular part of $f(z)$ at b_k

issue: There could be infinitely many poles. Convergence of $\sum_k P_k\left(\frac{1}{z-b_k}\right)$?

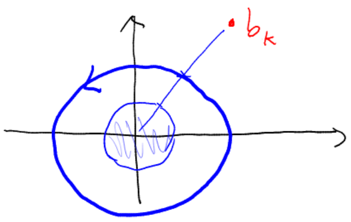
input: $\begin{cases} \{b_k\}_{k=1}^{\infty}: \text{a sequence of points in } \mathbb{C}, \quad \lim_{k\to\infty} b_k = \infty \\ P_k\left(\frac{1}{z-b_k}\right): \text{the singular part at } b_k \quad (P_k \text{ is a polynomial without constant term}) \end{cases}$

Try to correct $P_k\left(\frac{1}{z-b_k}\right)$ by a polynomial $g_k(z)$



so that $\sum_{k=1}^{\infty}\left(P_k\left(\frac{1}{z-b_k}\right) - g_k(z)\right)$ converges on any $B(0;R)$

But on $B(0;R)$ $\quad P_k\left(\frac{1}{z-b_k}\right)$ is analytic except for finitely many k



For each $b_k \neq 0$, let $g_k(z)$ be the Taylor expansion of $P_k\left(\frac{1}{z-b_k}\right)$ at 0, up to z^{n_k} ← chosen later

$$\left|P_k\left(\frac{1}{z-b_k}\right) - g_k(z)\right| \le |z|^{n_k+1} \int_{\partial B(0,\frac{|b_k|}{2})} \frac{|P_k(1/(\zeta-b_k))|}{|\zeta|^{n_k}|\zeta - z|} |d\zeta| \le M_k\left(\frac{2|z|}{|b_k|}\right)^{n_k+1}$$

$$\le M_k 2^{-(n_k+1)} \quad \text{for } |z| < \frac{|b_k|}{4} \qquad \left(M_k = 2\pi |b_k| \sup_{|\zeta| = |b_k|/2} \left|P_k\left(\frac{1}{\zeta - b_k}\right)\right|\right)$$

Choose $n_k \gg 1$ so that $M_k 2^{-(n_k+1)} < 2^{-k}$

$\Rightarrow$ For $z \in B(0;R)$, $\sum_{k=1}^{\infty}\left(P_k\left(\frac{1}{z-b_k}\right) - g_k(z)\right)$

$$= \underbrace{\sum_{|b_k|\le 4R}\left(P_k\left(\frac{1}{z-b_k}\right) - g_k(z)\right)}_{\text{finite sum}} + \underbrace{\sum_{|b_k|>4R}\left(P_k\left(\frac{1}{z-b_k}\right) - g_k(z)\right)}_{\text{converges absolutely}}$$

thm (Mittag-Leffler) Given input, there exist meromorphic functions whose poles are exactly $\{b_k\}$ with singular parts $P_k\left(\frac{1}{z-b_k}\right)$

Moreover, such a meromorphic function $f(z)$ can be written as

$$f(z) = \underbrace{\sum_k \left(P_k\left(\tfrac{1}{z-b_k}\right) - \underbrace{g_k(z)}_{\text{polynomial}}\right)}_{\text{converges absolutely on any compact subsets}} + \underbrace{g(z)}_{\text{entire}}$$

rmk clearly NOT unique

eg. 1. Consider $f(z) = \frac{\pi^2}{\sin^2(\pi z)}$ pole of order 2 at $n \in \mathbb{Z}$, $P_n\left(\frac{1}{z-n}\right) = \frac{1}{(z-n)^2}$

Also observe that $\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ converges uniformly on any compact subset

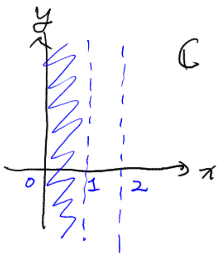
$$\Rightarrow \frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + \underbrace{g(z)}_{\text{entire}}$$

To determine $g(z)$, note that both $f(z)$ and $\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}$ are 1-periodic.

$z = x+iy$ $\qquad \sin \pi z = \frac{1}{2i}\left(e^{i\pi z} - e^{-i\pi z}\right)$

$$= \frac{1}{2i}\left(e^{-\pi y}(\cos \pi x + i \sin \pi x) - e^{\pi y}(\cos \pi x - i \sin \pi x)\right)$$

$$= \sin \pi x \cosh \pi y + i \cos \pi x \sinh \pi y$$



$$\Rightarrow |\sin \pi z|^2 = \sin^2 \pi x \cosh^2 \pi y + \cos^2 \pi x \sinh^2 \pi y$$

$$= (1-\cos^2\pi x)\cosh^2 \pi y + \cos^2 \pi x(\cosh^2 \pi y - 1) = \cosh^2 \pi y - \cos^2 \pi x$$

$$\Rightarrow \lim_{y \to \pm\infty} \frac{\pi^2}{\sin^2 \pi z} = 0 \quad \text{as } |y| \to \infty$$

Meanwhile, $\frac{1}{|z-n|^2} = \frac{1}{|x-n|^2 + y^2}$ $\Rightarrow$ $\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \to 0$ as $|y| \to \infty$

With the 1-periodicity $\Rightarrow$ $g(z)$ is a bounded entire function

By Liouville $\Rightarrow$ $g(z) = \text{constant} \overset{|y|\to\infty}{\Rightarrow} g(z) \equiv 0$

eg. 2 $\frac{\pi}{\sin \pi z}$: pole of order 1 at $n \in \mathbb{Z}$, with singular part $\frac{(-1)^n}{z-n}$

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{(-1)^n}{z-n} + \frac{(-1)^n}{n}\right) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n\left(\frac{1}{z-n} + \frac{1}{n}\right)$$ converges uniformly on compact subsets

$\boxed{\text{check}}$ $\left|\frac{1}{z-n} + \frac{1}{n}\right| = \frac{|z|}{|n||n-z|} < \frac{2R}{n^2}$ $\qquad$ if $|z| < R$, $|n| > 2R$, $|n-z| > |n| - |z| > \frac{|n|}{2}$

$$\Rightarrow \frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n\left(\frac{1}{z-n} + \frac{1}{n}\right) + g(z)$$

By the similar argument, $g(z) \equiv 0$

$$\Rightarrow \frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n=0}^{\infty} (-1)^n\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

§2 infinite products [A, §2.2 of ch. 5]

goal construct / understand entire functions with prescribed zeros

Start with infinite products of scalars.

$P = \prod_{k=1}^{\infty} c_k$ what does it mean ?

- We always exclude $P = 0$
- $P = \lim_{n\to\infty} (c_1 c_2 \cdots c_n)$
- In general, allow finitely many c_k's are zero and consider the product of nonzero ones.
- Let $P_n = c_1 c_2 \cdots c_n$. Since $P_n \xrightarrow{n\to\infty} P \neq 0$, $c_n = \frac{P_n}{P_{n-1}} \xrightarrow{n\to\infty} 1$
- Write $c_k = 1 + a_k \neq 0$

$$\prod_{k=1}^{\infty} c_k = \prod_{k=1}^{\infty} (1+a_k) = \prod_{k=1}^{\infty} \exp(\log(1+a_k)) = \exp\Big(\sum_{k=1}^{\infty} \log(1+a_k)\Big)$$

$\arg \in (-\pi, \pi)$

If $\prod_{k=1}^{\infty} c_k$ exists, $1 + a_k \to 1$; we may use the principal branch of log for sufficiently large k $\xRightarrow[\text{hard}]{\text{not}}$ $\sum_{k=1}^{\infty} \log(1+a_k)$ converges

- It is easy to see that $\sum_{k=1}^{\infty} \log(1+a_k)$ converges $\Rightarrow$ $\prod_{k=1}^{\infty} (1+a_k)$ converges.

  principal branch

- absolute convergence of $\sum_{k=1}^{\infty} \log(1+a_k)$?

  $\log(1+z) = z + z^2(\cdots)$ near $z=0$ $\Rightarrow$ if so, $a_k \to 0$

  $\Rightarrow$ $\frac{1}{2}|z| < |\log(1+z)| < 2|z|$ for $|z| < 1/1000$

  Hence, $\sum_{k=1}^{\infty} |\log(1+a_k)|$ converges if and only if $\sum_{k=1}^{\infty} |a_k|$ converges

§3 entire function with prescribed zeros [A, §2.3 of ch. 5]

Given $a_n \in \mathbb{C}$ with $a_n \to \infty$ as $n \to \infty$, $\exists$ entire function whose zeros are exactly a_n ?

a_n could be the same as a_{n+1}

- pull out zero, may assume $a_n \neq 0$ $\forall n$

  $\Rightarrow f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} (1 - \frac{z}{a_n})$ converges ?

  any entire function

  By the previous discussion, converges absolutely on any compact subset if and only if $\sum_{n=1}^{\infty} 1/|a_n| < \infty$

- Usually, it is not. For instance, $\{a_n\} = \{\pm 1, \pm 2, \ldots, \pm n, \ldots\}$
- Try to correct it: $\prod_{n=1}^{\infty} (1 - \frac{z}{a_n}) e^{P_n(z)}$ by some polynomials $P_n(z)$

  (compare it with the meromorphic function construction of Mittag-Leffler)

$$\overset{\log}{\rightsquigarrow} \quad \sum_{n=1}^{\infty} \left(\log\left(1-\frac{z}{a_n}\right) + P_n(z) \right)$$

- Since $a_n \to \infty$, $\left|\frac{z}{a_n}\right| < \frac{1}{2}$ for all except finitely many n when $|z|<R$

$$\log\left(1-\frac{z}{a_n}\right) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \cdots - \frac{1}{k}\left(\frac{z}{a_n}\right)^k - \cdots$$

$$\Rightarrow \log\left(1-\frac{z}{a_n}\right) + \underbrace{\sum_{k=1}^{m_n} \frac{1}{k}\left(\frac{z}{a_n}\right)^k}_{P_n(z)} = \frac{1}{m_n+1}\left(\frac{z}{a_n}\right)^{m_n+1} + \frac{1}{m_n+2}\left(\frac{z}{a_n}\right)^{m_n+2}$$

m_n: to be chosen later

$$\Rightarrow \left|\log\left(1-\frac{z}{a_n}\right) + P_n(z)\right| \le \frac{1}{m_n+1}\left(\frac{|z|}{|a_n|}\right)^{m_n+1}\left(1-\frac{|z|}{|a_n|}\right)^{-1}$$

$$< \boxed{\frac{2}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1}} < \frac{2}{m_n+1}\left(\frac{1}{2}\right)^{m_n+1}$$

$\Rightarrow$ If we set $m_n = n$, converges.

<u>thm</u> (Weierstrass) Every entire function with zeros exactly a_n (and zero of order m) can be written in the form

$$f(z) = z^m \underbrace{e^{g(z)}}_{\text{entire}} \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}\right)$$

certain integer to make the product converge.

choice of m_n: not unique

best choice?

<u>Cor</u> a meromorphic function on $\mathbb{C}$ is the quotient of two entire functions.

Same degree here

Canonical product Can m_n be taken to be the same?

Namely, $\exists\, h \in \mathbb{N}$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{h+1}\left(\frac{R}{|a_n|}\right)^{h+1}$ converges $\forall R$

$$= \frac{R^{h+1}}{h+1}\sum_{n=1}^{\infty}\left(\frac{1}{|a_n|}\right)^{h+1}$$

<u>defn</u> If so, let h be the smallest integer so that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{|a_n|}\right)^{h+1} < \infty$

$\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n} + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h\right)$ is called the <u>canonical</u> <u>product</u> of $\{a_n\}$ and h is called its <u>genus</u>

<u>e.g.</u> $a_n = n$, $h = 1$

<u>defn</u> In this case, any analytic function with zeros $\{0,\dots,0\} \cup \{a_n\}$ is $\displaystyle f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)\exp\left(\frac{z}{a_n} + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h\right)$

If $g(z)$ is a polynomial, $f(z)$ is said to have <u>finite</u> <u>genus</u> and its genus is defined to be $\max\{\deg g(z), h\}$

Up to factor zeros and log, NOT transcendental.

<u>e.g.</u> $\displaystyle \sin \pi z = \pi z \prod_{n \neq 0}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}$ has genus 1.

$$= \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

§ 4 Gamma function [A, § 2.4 of ch. 5]

We can split the canonical product of $\sin \pi z$ into positive and negative integers: $G(z) = \prod_{n=1}^{\infty} (1+\frac{z}{n}) e^{-\frac{z}{n}}$ has zeros on $-\mathbb{N}$

- $\sin \pi z = \pi \, G(z) \, G(-z)$
- NOT 1-periodic, but $G(z-1)$ has zeros on $\{0\} \cup \{-\mathbb{N}\}$

  Hence, $G(z-1) = z e^{\gamma(z)} G(z)$

  Consider $(\log(-))'$ of both sides:

$$\sum_{n=1}^{\infty} \left(\frac{1}{z-1+n} - \frac{1}{n}\right) = \frac{1}{z} + \gamma'(z) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right) \Rightarrow \gamma'(z) = 0$$

$$= \frac{1}{z} - 1 + \sum_{n=2}^{\infty} \left(\frac{1}{z-1+n} - \frac{1}{n}\right)$$

$$= \frac{1}{z} - 1 + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} - \frac{1}{m+1}\right) = \frac{1}{z} - 1 + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} - \frac{1}{m} + \underbrace{\frac{1}{m} - \frac{1}{m+1}}_{=1}\right)$$

At $z=1$, $e^{\gamma} = G(1) / G(0) = \prod_{n=1}^{\infty} (1+\frac{1}{n}) e^{-\frac{1}{n}}$

$$= \exp\left(\sum_{n=1}^{\infty} \left(\log(n+1) - \log n - \frac{1}{n}\right)\right)$$

$\Rightarrow \gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$: Euler's constant $\Rightarrow G(1) = e^{-\gamma}$

- Now, $G(z-1) = z e^{\gamma} G(z)$

$$\Rightarrow e^{\gamma(z-1)} G(z-1) = z \, e^{\gamma z} G(z) \Rightarrow H(z-1) = z \underbrace{H(z)}_{=\, e^{\gamma z} \prod_{n=1}^{\infty}(1+\frac{z}{n}) e^{-\frac{z}{n}}}$$

$$\Rightarrow H(z) = (z+1) H(z+1) \Rightarrow \frac{1}{H(z+1)} = (z+1) \frac{1}{H(z)} \Rightarrow \frac{1}{(z+1)\, H(z+1)} = \frac{z}{z H(z)}$$

<u>goal</u> define a function $\Gamma(z)$ whose value for n is $(n-1)$!

<u>attempt</u> $\Gamma(z+1) = z \, \Gamma(z)$ with $\Gamma(1) = 1$

<u>defn</u> $\Gamma(z) = \frac{1}{z H(z)} = \left(z e^{\gamma z} G(z)\right)^{-1} = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} (1+\frac{z}{n})^{-1} e^{\frac{z}{n}}$

$$\Gamma(1) = \left(e^{\gamma} G(1)\right)^{-1} = 1$$

basic functional equations

i) $\sin \pi z = \pi z \, G(z) \, G(-z)$

$$\Rightarrow \frac{\pi}{\sin \pi z} = \frac{1}{z} \frac{1}{G(z)} \frac{1}{G(-z)} = \frac{1}{z} \cancel{z} \cancel{e^{\gamma z}} \Gamma(z) (-z) \cancel{e^{-\gamma z}} \Gamma(-z) = \Gamma(z) \, \Gamma(1-z)$$

At $z = \frac{1}{2}$, $(\Gamma(\frac{1}{2}))^2 = \pi \Rightarrow \Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$ (nonnegative from the defining equation)

ii) $\Gamma(z)$ has no zeros, and has poles of order 1 at $\{0, -1, -2, \cdots\}$

$\boxed{\text{check}}$ $\operatorname{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}$

$$\left[\Gamma(z) = \frac{\pi}{\sin \pi z} \frac{1}{\Gamma(1-z)} = \left(\frac{(-1)^n}{(z+n)} + \cdots\right) \left(\frac{1}{n!} + (z+n)(\cdots)\right) \right]$$

iii) Consider the $(\log \Gamma(z))' = \dfrac{\Gamma'(z)}{\Gamma(z)}$

$\Gamma(z)$ (or $G(z)$) is constructed by product; $(\log \Gamma(z))'$ would have a summation form

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \sum_{n=1}^{\infty}\left(-\left(1+\frac{z}{n}\right)^{-1}\frac{1}{n} + \frac{1}{n}\right)$$

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \frac{1}{z^2} + \sum_{n=1}^{\infty}\frac{1}{(z+n)^2} = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}$$

on $R = \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\}$
- traditional expression only works on R
- will need $\log z$
- $\{0, -1, \cdots, -n \cdots\}$ bad for $\Gamma(z)$
- iR: bad for the argument below

§5 Stirling's formula [A, §2.5 of ch. 5]

goal obtain an integral expression of $\Gamma(z)$

Starting point: $\dfrac{d}{dz}\left(\dfrac{\Gamma'(z)}{\Gamma(z)}\right) = \displaystyle\sum_{n=0}^{\infty}\frac{1}{z^2+n^2}$

1° We may think $\dfrac{1}{(z+n)^2}$ as residues (another application of residue: to evaluate summations)

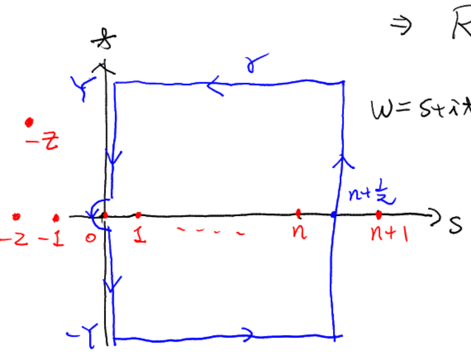
Fix $z \notin \{0, -1, \cdots, -n, \cdots\}$

$\dfrac{1}{(z+n)^2}$: value of $\dfrac{1}{(z+w)^2}$ at $w=n$ ⇝ $\sin(\pi w)$ has zeros at $\mathbb{Z}$

$$\frac{\pi}{\sin \pi w} = \frac{(-1)^n}{\sin \pi w}$$

Consider $\Phi(w) = \dfrac{1}{(z+w)^2}\,\dfrac{\pi \cos \pi w}{\sin \pi w}$ ⇝ correct it by $\dfrac{\pi \cos \pi w}{\sin \pi w} = \dfrac{g'}{g}$

$\Rightarrow \mathrm{Res}_{w=n}\,\Phi(w) = \dfrac{1}{(z+n)^2}$ for n = non-negative integer



$w = s+it$

$$\boxed{\begin{aligned} \cos \pi w &= \cos \pi s \cosh \pi t - i \sin \pi s \sinh \pi t \\ \sin \pi w &= \sin \pi s \cosh \pi t + i \cos \pi s \sinh \pi t \end{aligned} \quad w = s+it}$$

$$\frac{1}{2\pi i}\int_\gamma \Phi(w)\, dw = \sum_{k=0}^{n}\frac{1}{(z+k)^2} \quad (\text{for } z \notin iR)$$

[top/bottom] $\dfrac{\cos \pi w}{\sin \pi w} \to \mp i$ as $Y \to \infty$

$$\left|\frac{1}{2\pi i}\int_{\text{top}} \Phi(w)\, dw\right| \le \int_0^{n+\frac{1}{2}} \frac{1}{|z+s+iY|^2}\, ds \to 0 \quad \text{as } Y \to \infty$$

[right] $\dfrac{\cos \pi w}{\sin \pi w}$ is uniformly bounded on $s = n+\frac{1}{2}$ (and the bound is independent of n)

$$\left|\frac{1}{2\pi i}\int_{\text{right}} \Phi(w)\, dw\right| \le 100\int_{-\infty}^{\infty}\frac{1}{|z+n+\frac{1}{2}+it|^2}\, dt \to 0 \quad \text{as } n \to \infty$$

$$\Big\| \quad \int_{-\infty}^{\infty}\frac{1}{t^2+(n+\frac{1}{2}+\mathrm{Re}\, z)^2}\, dt = \frac{\pi}{n+\frac{1}{2}+\mathrm{Re}\, z}$$

[small circle] near $w=0$

$$\Phi(w) = \frac{1}{z^2}\frac{1}{w} + (\text{regular in } w)$$

$$\frac{1}{2\pi i}\int_\epsilon \Phi(w)\, dw = \frac{1}{2z^2} + \underset{\substack{\downarrow \\ 0}}{(\text{proportional to radius})} \quad \text{as radius} \to 0$$

[left] $\quad x=0 \quad \frac{\cos \pi w}{\sin \pi w} \overset{w=it}{=} -i \coth \pi t$

orientation $\quad \leftarrow w=it \qquad \leftarrow w=-it$

$$-\frac{1}{2\pi i}\int_0^\infty (-i\pi\coth\pi t)\,\frac{1}{(z+it)^2}\, i\,dt + \frac{1}{2\pi i}\int_0^\infty (i\pi\coth\pi t)\,\frac{1}{(z-it)^2}\,(-i\,dt)$$

$$= \frac{1}{2}\int_0^\infty i\coth\pi t\left(\frac{1}{(z+it)^2}-\frac{1}{(z-it)^2}\right)dt = \int_0^\infty \coth\pi t\,\frac{2zt}{(z^2+t^2)^2}\,dt$$

Hence, $\boxed{\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \frac{1}{2z^2} + \int_0^\infty \coth\pi t\,\frac{2zt}{(z^2+t^2)^2}\,dt}$

2° Now, try to evaluate the integral, or integrate it against dz

$$\coth\pi z = \frac{\cosh\pi z}{\sinh\pi z} = \frac{e^{\pi z}+e^{-\pi z}}{e^{\pi z}-e^{-\pi z}} = \frac{e^{2\pi z}+1}{e^{2\pi z}-1} = 1 + \frac{2}{e^{2\pi z}-1} \longrightarrow \text{small for } t\to\infty$$

$$\int_0^\infty \frac{2zt}{(z^2+t^2)^2}\,dt = \int_0^\infty \frac{z}{(s+z^2)^2}\,ds = \frac{1}{z}$$

$$\Rightarrow \frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \frac{1}{z} + \frac{1}{2z^2} + \int_0^\infty \frac{4zt}{(z^2+t^2)^2}\,\frac{dt}{e^{2\pi t}-1}$$

$$\Rightarrow \frac{\Gamma'(z)}{\Gamma(z)} = C + \log z - \frac{1}{2z} - \int_0^\infty \frac{2t}{z^2+t^2}\,\frac{dt}{e^{2\pi t}-1}$$

- choose the principal branch of $\log$, and $z \notin \mathbb{R}_{\le 0}$
- trouble: $z^2+t^2=0 \Rightarrow z \in i\mathbb{R}$

When $\operatorname{Re} z > 0 \overset{\text{Check}}{\Longrightarrow}$ uniform convergence of both integral as a function of z ($\in$ any compact set)

- Also, $\Gamma(z)$ is better behaved when $\operatorname{Re} z>0$. ($\rightsquigarrow \log\Gamma(z)$ is well-defined)

More precisely, it is a nowhere zero analytic function there.

3° $\log\Gamma(z) = ?$

$\frac{2t}{z^2+t^2} \rightsquigarrow \arctan\frac{z}{t}$: NO good

To get rid of arctan, perform integration by parts

$$-\int_0^\infty \frac{2t}{z^2+t^2}\,\frac{dt}{e^{2\pi t}-1} = -\int_0^\infty \frac{2t}{z^2+t^2}\,\frac{\partial}{\partial t}\left(\frac{1}{2\pi}\log(1-e^{-2\pi t})\right)dt$$

$$= \frac{1}{\pi}\int_0^\infty \frac{z^2-t^2}{(z^2+t^2)^2}\log(1-e^{-2\pi t})\,dt$$

$\frac{\partial}{\partial t}\left(\frac{1}{2\pi}\log(1-e^{-2\pi t})\right) = \frac{e^{-2\pi t}}{1-e^{-2\pi t}} = \frac{1}{e^{2\pi t}-1}$

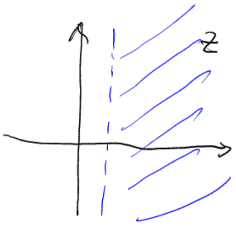
$\frac{\partial}{\partial t}\frac{t}{z^2+t^2} = \frac{z^2-t^2}{(z^2+t^2)^2}$

$\frac{\partial}{\partial z}\frac{z}{z^2+t^2} = \frac{t^2-z^2}{(z^2+t^2)^2}$

$$\Rightarrow \log\Gamma(z) = C' + Cz + z\log z - z - \frac{1}{2}\log z - \frac{1}{\pi}\int_0^\infty \frac{z}{z^2+t^2}\log(1-e^{-2\pi t})\,dt$$

4° Let $\quad J(z) = \frac{1}{\pi}\int_0^\infty \frac{z}{z^2+t^2}\log\frac{1}{1-e^{-2\pi t}}\,dt$

What are C' and C? What is $J(z)$ as $z\to\infty$?



$x=c$

$|z^2+t^2| > |z|^2 - |t|^2 \sim O(|z|^2)$ for t: small, say $|t| < \frac{|z|}{2}$

$$|z^2+t^2| = \underset{x+i(y+t)}{|z+it|}\;\underset{x+i(y-t)}{|z-it|} \ge \begin{cases} |z|\,c & \text{when } \operatorname{Im} z>0 \\ c\,|z| & \text{when } \operatorname{Im} z<0 \end{cases}$$

when $\operatorname{Re} z \ge c > 0$

NO useful lower bound near the imaginary axis

$$\Rightarrow |\pi J(z)| \le \left|\int_0^{\frac{|z|}{2}}\cdots\right| + \left|\int_{\frac{|z|}{2}}^\infty\cdots\right| \le \frac{100}{|z|}\int_0^\infty \log\frac{1}{1-e^{-2\pi t}}\,dt + \frac{100}{c}\int_{|z|/2}^\infty \log\frac{1}{1-e^{-2\pi t}}\,dt$$

$\left(\begin{array}{ll} \text{for } t\sim 0 : \frac{1}{1-e^{-2\pi t}} \sim \frac{1}{2\pi t} & \int \log t \, dt = t\log t - t + \text{const.} \\ \text{for } t \gg 1 : \frac{1}{1-e^{-2\pi t}} \sim 1 + e^{-2\pi t} & \log(1+e^{-2\pi t}) \sim e^{-2\pi t} : \text{integrable} \end{array} \right)$

Thus, $J(z) \to 0$ as $z \to \infty$ (away from the y-axis)

5° $\Gamma(z+1) = z\,\Gamma(z)$

$\log \Gamma(z+1) = \log z + \log \Gamma(z)$

$$C' + (C-1)(z+1) + (z+1)\log(z+1) - \tfrac{1}{2}\log(z+1) + J(z+1)$$
$$= \log z + C' + (C-1)z + z\log z - \tfrac{1}{2}\log z + J(z)$$

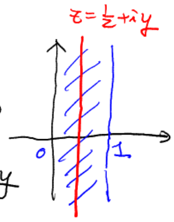
$$\Rightarrow \quad C-1 = (z+\tfrac{1}{2})\underbrace{\left(\log z - \log(z+1)\right)}_{\shortparallel} + J(z) - J(z+1)$$

$$-\log(1+\tfrac{1}{z}) = -\left(\tfrac{1}{z} + O(\tfrac{1}{z^2})\right)$$

As $z \to \infty \quad \Rightarrow \boxed{C = 0}$

6° $\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ need $\mathrm{Re}\, z > 0$ and $\mathrm{Re}\, 1-z > 0 \Rightarrow$

for z & $1-z \in R$, Consider $z = \frac{1}{2} + iy$



$$\log \Gamma(\tfrac{1}{2}+iy) + \log \Gamma(\tfrac{1}{2}-iy) = \log \frac{\pi}{\cosh \pi y}$$

$$2C' - (\tfrac{1}{2}+iy) - (\tfrac{1}{2}-iy) + iy\log(\tfrac{1}{2}+iy) - iy\log(\tfrac{1}{2}-iy) + J(\tfrac{1}{2}+iy) + J(\tfrac{1}{2}-iy)$$
$$= \log \pi - \log \cosh \pi y$$

When $y \gg 1$

(same result for $y \to -\infty$)

$$iy\log(\tfrac{1}{2}+iy) = iy\left(\log(1+\tfrac{1}{2iy}) - \log\tfrac{1}{iy}\right)$$
$$= iy\left(\tfrac{1}{2iy} + O(\tfrac{1}{y^2}) + \log y + i\tfrac{\pi}{2}\right)$$

$$iy\log(\tfrac{1}{2}-iy) = iy\left(\log(1-\tfrac{1}{2iy}) - \log\tfrac{-1}{iy}\right)$$
$$= iy\left(-\tfrac{1}{2iy} + O(\tfrac{1}{y^2}) + \log y - i\tfrac{\pi}{2}\right)$$

$$\log\cosh \pi y = \log \frac{e^{\pi y}+e^{-\pi y}}{2} = \log e^{\pi y} + \log(1-e^{-2\pi y}) - \log 2$$
$$= \pi y + O(e^{-2\pi y}) - \log 2$$

$$\Rightarrow \quad 2C' - 1 + 1 - \pi y + O(\tfrac{1}{y}) = \log \pi - \pi y + \log 2 - O(e^{-2\pi y})$$

$$\Rightarrow \quad \boxed{C' = \tfrac{1}{2}\log 2\pi}$$

<u>Upshot</u>

$$\boxed{\begin{array}{l} \log \Gamma(z) = \frac{1}{2}\log 2\pi + (z-\frac{1}{2})\log z - z + J(z) \\ \Rightarrow \Gamma(z) = \sqrt{2\pi}\; z^{z-\frac{1}{2}}\; e^{-z}\; e^{J(z)} \end{array}}$$

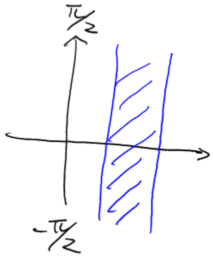
$e^{J(z)} \to 1$ as $|z| \to \infty$ away from $i\mathbb{R}$

for $\mathrm{Re}\, z > 0$

<u>Cor</u> $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$ for $\mathrm{Re}\, z > 0$

pf: $\boxed{\text{check}}$ $\left\{ \begin{array}{l} \text{right hand side } (\tilde{\Gamma}(z)) \text{ defines an analytic function when } \mathrm{Re}\, z > 0 \\ \tilde{\Gamma}(z+1) = z\,\tilde{\Gamma}(z) \quad : \text{integration by parts} \end{array} \right.$

$$\Rightarrow \quad \frac{\tilde\Gamma(z+1)}{\Gamma(z+1)} = \frac{z\tilde\Gamma(z)}{z\Gamma(z)} = \frac{\tilde\Gamma(z)}{\Gamma(z)} \quad : \text{ 1-periodic } \underline{\text{entire}} \text{ function!}$$



Consider them on the strip $S = \{ z \mid 1 \le \operatorname{Re} z \le 2 \}$

$$|\tilde\Gamma(z)| \le \int_0^\infty e^{-t} t^{x-1}\, dt = \Gamma(x) \quad : \text{bounded on } S$$

$x = \operatorname{Re} z \qquad |t^{z-1}| = t^{x-1} |t^{iy}| = t^{x-1}$

$$\log|\Gamma(z)| = \operatorname{Re}(\log \Gamma(z))$$

$$= \frac{1}{2}\log 2\pi + \underbrace{\left(x - \tfrac{1}{2}\right)}_{\text{bdd}} \underbrace{\log|z|}_{\sim \log|y|} - \underbrace{y \arg z}_{\sim \frac{\pi}{2}|y|} - \underbrace{x}_{\text{bdd}} + \operatorname{Re} \underbrace{J(z)}_{\to 0 \; (y \to \infty)}$$

$$\Rightarrow \quad \left| \frac{\tilde\Gamma(z)}{\Gamma(z)} \right| \le \tilde C e^{+\frac{\pi}{2}|y|} \quad \text{on } S$$

By 1-periodicity, estimate holds for $z \in \mathbb{C}$

NOT enough to show that $\dfrac{\tilde\Gamma(z)}{\Gamma(z)}$ is constant.

$y = \operatorname{Im} z = -\frac{1}{2\pi} \log|\zeta|$

For 1-periodic function, introduce $\zeta = e^{2\pi i z}$

$$\Rightarrow \quad h(\zeta) = \frac{\tilde\Gamma(\log\zeta / 2\pi i)}{\Gamma(\log\zeta / 2\pi i)} \quad \text{is an analytic function for } \zeta \in \mathbb{C} \setminus \{0\}$$

with
$$|h(\zeta)| \le \begin{cases} \tilde C e^{-\frac{1}{4}|\zeta|} = \tilde C |\zeta|^{-\frac{1}{4}} & \text{for } |\zeta| \ll 1 \\ \tilde C e^{\frac{1}{4}|\zeta|} = \tilde C |\zeta|^{\frac{1}{4}} & \text{for } |\zeta| \gg 1 \end{cases}$$

$$|y| = \begin{cases} -\frac{1}{2\pi}\log|\zeta| & |\zeta| < 1 \\ \frac{1}{2\pi}\log|\zeta| & |\zeta| > 1 \end{cases}$$

$\zeta = 0$ is a removable singularity

enough for Liouville's estimate

$\Rightarrow h = \text{constant}$

$$\Rightarrow \quad \frac{\tilde\Gamma(z)}{\Gamma(z)} = \frac{\tilde\Gamma(1)}{\Gamma(1)} = \frac{1}{1} \qquad \#$$

(or, consider $h(1/\xi)$ near $\xi = 0$
$\Rightarrow$ removable singularity)