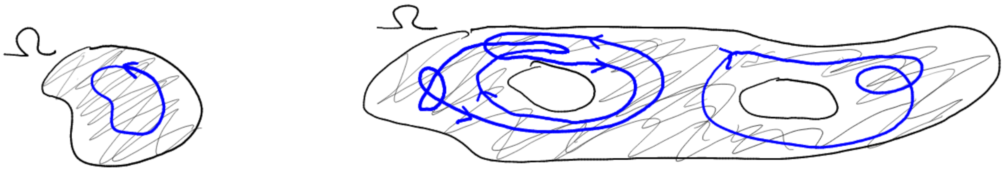
V. residue and argument principle

§1 general form of Cauchy's theorem [Ahlfors, §4 of ch.4]

We will NOT do the detail of this part, and will just point out some key concepts here.

Ω: open and connected. γ : closed curve in Ω (with direction)

$f(z)$: analytic in Ω $\quad \int_\gamma f\,dz \overset{?}{=} 0$



explain via pictures

<u>defn</u> Ω is called "simply-connected"

if $\hat{\mathbb{C}} \setminus \Omega$ is connected

$\Leftrightarrow$ $n(\gamma, a) = 0 \quad \forall\ \gamma \subset \Omega$ and $a \notin \Omega$

(basically "no holes" in Ω)

<u>defn</u> $\gamma \subset \Omega$ is homologous to zero with respect to Ω

if $n(\gamma, a) = 0 \quad \forall\ a \notin \Omega$

(no obstruction for γ to shrink to a point in Ω)

<u>thm</u> $f(z)$: analytic in Ω $\Rightarrow$ $\int_\gamma f(z)\,dz = 0$

for any γ which is homologous to zero with respect to Ω

(since γ can be shrinked to a point in Ω,
γ is more or less the boundary of some region
on where f is analytic)

<u>Cor</u> Ω: simply-connected , $f(z)$: analytic and nowhere zero
in Ω $\Rightarrow$ $\log f(z)$ and $(f(z))^{\frac{1}{n}}$ can be defined

pf: By simply-connectedness, $\int_\gamma \frac{f'}{f}\,dz = 0 \quad \forall\ \gamma \subset \Omega$

$\Rightarrow \frac{f'}{f} = u'$ for some u: analytic ※

※ This part will NOT be included in the midterm or final.

※ shrink to a point $\leftrightsquigarrow$ boundary of something
are DIFFERENT concepts in higher
dimension or on surface (with topology)
The notion in Ahlfors in a short cut for regions in $\mathbb{C}$

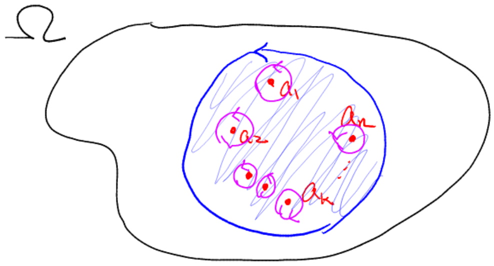
§2 the residue theorem [Ahlfors, §5.1 of ch.4]

goal $f(z)$: analytic on $\Omega \setminus \{$points with no accumulation point in $\Omega\}$

In other words, $f(z)$ has only isolated singularities on Ω

Compute $\int_\gamma f(z)\,dz$, γ : not passing through singularities

Consider the simplest case : $\gamma = \partial R$, $R \subset \Omega$



Since $\bar{R}$ is compact, there are only finite number of singularities in R, $\{a_1, \cdots, a_n\}$

For each a_k choose $\varepsilon_k > 0$, such that $\overline{B(a_k, \varepsilon_k)} \subset R$ and pairwise disjoint.

Let $\sigma_k = \partial B(a_k, \varepsilon_k)$.

By Cauchy theorem (for $f(z)$ on $\bar{R} \setminus \bigcup_{k=1}^{n} B(a_k, \varepsilon_k)$)

$$\int_\gamma f(z)\,dz - \sum_{k=1}^{n} \int_{\sigma_k} f(z)\,dz = 0$$

(the minus sign is) due to orientation

Hence, we only need to compute $\int_{\sigma_k} f(z)\,dz$

As explained last week, $f(z)$ has a Laurent series expansion on a punctured disk at a_k. and the convergence is uniform on any compact subset $f(z) = \sum_{l=1}^{n} B_{-l}(z-a_k)^{-l} + \sum_{l=0}^{n} B_l (z-a_k)^l$

$$\Rightarrow \quad f(z) - B_{-1}(z-a_k) = \frac{d}{dz}\left(\sum_{l=2}^{n} \frac{B_{-l}}{-l+1}(z-a_k)^{-l+1} + \sum_{l=0}^{n} \frac{B_l}{l+1}(z-a_k)^{l+1} \right)$$

$$\Rightarrow \quad \int_{\sigma_k} f(z)\,dz = \int_{\sigma_k} \frac{B_{-1}}{z-a_k}\,dz = 2\pi i\, B_{-1}$$

($\Rightarrow$ because) $\left(f(z) - \frac{B_{-1}}{z-a_k}\right) dz$ is an exact differential

To sum up, $\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} (B_{-1}$ at $a_k)$

def / thm $f(z)$ analytic on Ω except isolated singularities.

For an isolated singularity a, define $\mathrm{Res}_{z=a} f$ to be the coefficient of $(z-a)^{-1}$ in the Laurent series at a

Then, $\frac{1}{2\pi i} \int_\gamma f(z)\,dz = \sum_j n(\gamma; a_j)\, \mathrm{Res}_{z=a_j} f$

for each $\gamma \subset \Omega$ not passing through the singularities and homologous to zero sketch of the proof:

i) claim there are only finitely many singularities with $n(\sigma, a_j) \neq 0$

$Z = \{ b \in \mathbb{C} \setminus \{\gamma\} \mid n(\gamma, b) = 0 \}$ is open, and contains the unbounded component of $\mathbb{C} \setminus \{\gamma\}$ $\Rightarrow$ $\mathbb{C} \setminus Z$ is closed and bounded

Hence, $\mathbb{C} \setminus Z$ is compact,

Since the singularities are ISOLATED, there cannot be infinitely many of them in $\mathbb{C} \setminus Z$

ii) For each a_j, let $\sigma_j = \partial$(small disk around it)
↳ disjoint from other disks and also γ

$\Rightarrow$ $\gamma - \sum_j n(\sigma, a_j)\, \sigma_j$ is homologous to zero with respect to $\Omega' = \Omega \setminus \{\text{singularities}\}$

need γ is homologous to zero with respect to Ω

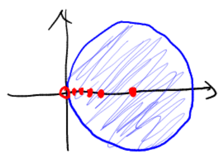
By Cauchy $\Rightarrow$ $\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_j \frac{n(\gamma, a_j)}{2\pi i} \int_{\sigma_j} f(z)\, dz$

$= \sum_j n(\sigma, a_j) \operatorname{Res}_{z=a_j} f$ ※

rmk possible picture of isolated singularities:

$\Omega = B(1;1)$, singularities $= \{ \frac{1}{k} \mid k \in \mathbb{N} \}$

0 = accumulation point, but NOT in Ω



$\gamma \subset \Omega$ Compactness imply that $n(\gamma, \frac{1}{k}) \neq 0$ for finitely many k

§3 argument principle [Ahlfors, §5.2 of ch.4]

notion an analytic on Ω except poles is called a meromorphic function (locally analytic or $\frac{1}{\text{analytic}}$)

recall $\frac{1}{2\pi i} \int_\gamma \frac{df}{f} = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f}\, dz$ is the number of zeros of f enclosed by γ

Now, for an meromorphic function $f(z)$ in Ω,

Consider $\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}\, dz$ over γ: homologous to zero with respect to Ω

discussion: singularities of $\frac{f'}{f}$ comes from the poles of f and the zeros of f

at a pole $z = b_k$, $f(z) = (z - b_k)^{-l_k} g(z)$ $\quad l_k \in \mathbb{N}$ is the order of b_k

$f'(z) = -l_k (z - b_k)^{-l_k - 1} g(z) + (z - b_k)^{-l_k} g'(z)$

$\Rightarrow \frac{f'}{f} = -\frac{l_k}{z - b_k} + \frac{g'(z)}{g(z)}$ → analytic near $z = b_k$

$\Rightarrow \operatorname{Res}_{z=b_k} \frac{f'}{f} = -l_k$

Similarly, at a zero $z = a_k$, $\operatorname{Res}_{z=a_j} \frac{f'}{f}$ = order of vanishing

By the residue theorem,

$$\Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

(repeat the times of the corresponding order)

- For simplicity, take $\gamma = \partial(\text{some ball / rectangle} \subset \Omega)$

$$n(f(\gamma), 0) = \frac{1}{2\pi i}\int_{\Gamma = f(\gamma)} \frac{dw}{w} = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \#\{\text{zeros enclosed by } \gamma\} - \#\{\text{poles enclosed by } \gamma\}$$

- It can be used to estimate the zeros by comparison trick:

g & h: analytic on Ω, γ = boundary of some region R

Consider $\frac{g}{h}$ (assume $g(\zeta) \neq 0$, $h(\zeta) \neq 0$ for $\zeta \in \gamma$)

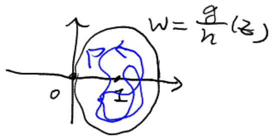
It is not hard to see that

$$n(\Gamma = \tfrac{g}{h}(\gamma), 0) = \#\{\text{zeros of } g/h \text{ in } R\} - \#\{\text{poles of } g/h \text{ in } R\}$$
$$= \#\{\text{zeros of } g \text{ in } R\} - \#\{\text{zeros of } h \text{ in } R\}$$

If $\frac{g}{h}(\gamma)$ has zero index around 0,

then g & h has the same number of zeros in R

For instance,



$\left|\frac{g}{h} - 1\right| < 1$ on γ

$\Leftrightarrow |g(z) - h(z)| < h(z) \quad \forall z \in \gamma$

__Cor__ (Rouché's thm) $\gamma = \partial R$, g, h: analytic on R

if $|g(z) - h(z)| < h(z) \quad \forall z \in \gamma$, then

g & h has the same number of zeros in R

__e.g.__ zeros of $g(z) = z^7 - 2z^5 + 6z^3 + z - 1$ with $|z| < 1$?

$|z| = 1$: $6z^3$ dominates $g(z)$ in the sense that

$$|g(z) - 6z^3| = |z^7 - 2z^5 + z - 1| \leq 5 < |6z^3| \quad \text{for } |z| = 1$$

$\Rightarrow g(z) = 0$ has 3 roots with $|z| < 1$

__Cor__ (limit of nowhere zero analytic functions, [Ahlfors, §1.1 of ch.5])

Ω: open & connected, $\{f_n\}$: analytic in Ω and nowhere zero

$f_n \to f$ in Ω and uniformly on compact subsets

Then, $f \equiv 0$ or f is nowhere zero.

Pf: If f is not identically zero, it can has only isolated zero.

$$\text{vanishing order at } a = \frac{1}{2\pi i}\int_{\partial B(a;\varepsilon)} \frac{f'}{f}\,dz = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial B(a;\varepsilon)} \frac{f_n'}{f_n}\,dz = 0$$

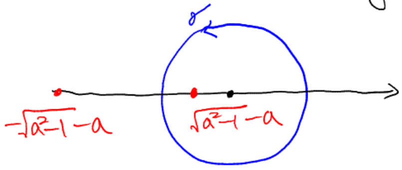
※

§4 evaluate integrals by residue theorem [Ahlfors, §5.3 of ch.4]

example 1 $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}$ $\quad a>1$

sol'n: For $z \in \gamma = \partial B(0,1)$ $\quad z = e^{i\theta}$ $\quad d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$



The integral $\displaystyle = \int_\gamma \frac{1}{(a+\frac{z}{2}+\frac{1}{2z})^2}\frac{dz}{iz}$

$\displaystyle = -4i \int_\gamma \frac{z}{(z^2+2az+1)^2}\,dz$

$\displaystyle = (-4i)(2\pi i)\ \mathrm{Res}_{z=\sqrt{a^2-1}-a} \frac{z}{(z^2+2az+1)^2}$

root of $z^2+2az+1=0$
$(z+a)^2 = a^2-1$
$z = \pm\sqrt{a^2-1}-a$

Let $u = \sqrt{a^2-1}-a$, $v = -\sqrt{a^2-1}-a$, $\quad f(z) = \frac{z}{(z-u)^2(z-v)^2}$ $\quad z=u$ is a pole of order 2

$\displaystyle \frac{z}{(z-v)^2} = \frac{u}{(u-v)^2} - \frac{u+v}{(u-v)^3}(z-v) + (z-v)^2 g(z)$

$\displaystyle \frac{d}{dz}\left(\frac{z}{(z-v)^2}\right) = \frac{1}{(z-v)^2} - \frac{2z}{(z-v)^3} = \frac{-(z+v)}{(z-v)^3}$ $\quad \Rightarrow \mathrm{Res}_{z=u} f(z) = \frac{-(u+v)}{(u-v)^3}$

$u+v = -2a$
$u-v = 2(a^2-1)^{\frac{1}{2}}$

$\boxed{?} = 8\pi \frac{2a}{8(a^2-1)^{3/2}} = \frac{2\pi a}{(a^2-1)^{3/2}}$

⋇

Basically, $\displaystyle\int_0^{2\pi}$ (rational function in $\cos\theta$ and $\sin\theta$) $d\theta$ can be evaluated this way.

example 2 $\displaystyle\int_0^\infty \frac{x^{-c}}{1+x}\,dx = \frac{\pi}{\sin\pi c}$ $\quad 0<c<1$

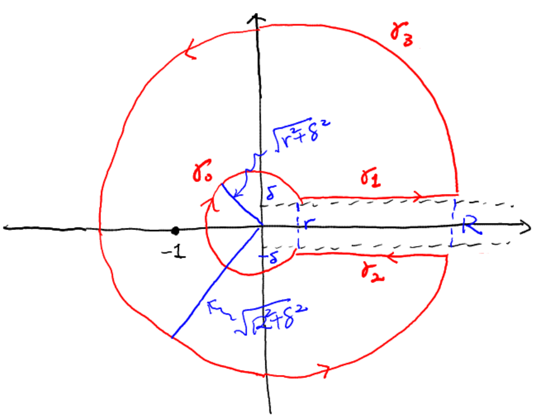
sol'n: • $z^{-c} = \exp(-c\log z)$: NOT single valued,
it is single valued on $\mathbb{C}\setminus\{\text{a ray}\}$

• evaluate on the positive real-axis

• $\frac{x^{-c}}{1+x} < x^{1-c}$ for $x>1$ $\Rightarrow \int_1^\infty x^{1-c}\,dx < \infty$ since $c>0$

• $\frac{x^{-c}}{1+x} < x^{-c}$ for $x<1$ $\Rightarrow \int_0^1 x^{-c}\,dx < \infty$ since $c<1$

⇝ Consider the line integral of $\frac{z^{-c}}{1+z}$ over the following curve, and let $r, \delta \to 0$, $R\to\infty$.



$r<\delta<R$

On $\mathbb{C}\setminus\{\text{positive x-axis}\}$, $\log z$ is well-defined.
Choose the branch so that $\arg z \in (0, 2\pi)$

$\gamma = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3$

$\displaystyle\int_\gamma \frac{z^{-c}}{1+z}\,dz = 2\pi i\ \mathrm{Res}_{z=-1}\frac{z^{-c}}{1+z} = 2\pi i \exp(-c\log(-1))$

$= 2\pi i \exp(-c\pi i)$

For γ_0 : $\left|\int_{\gamma_0} \frac{z^{-c}}{1+z} dz\right| \le \int_{\gamma_0} \frac{|z|^{-c}}{|1+z|} |dz| \le \frac{(\sqrt{r^2+\delta^2})^{-c}}{1-\sqrt{r^2+\delta^2}} 2\pi \sqrt{r^2+\delta^2} \xrightarrow{\text{as } r,\delta\to 0} 0$ ($\because c<1$)

For γ_3 : $\left|\int_{\gamma_3} \frac{z^{-c}}{1+z} dz\right| \le \int_{\gamma_3} \frac{|z|^{-c}}{|1+z|} |dz| \le \frac{(\sqrt{R^2+\delta^2})^{-c}}{\sqrt{R^2+\delta^2}-1} 2\pi \sqrt{R^2+\delta^2} \xrightarrow{\text{as } R\to\infty} 0$ ($\because c>0$)

For γ_1 : $z = x+i\delta \qquad r<x<R$ (let $\delta \to 0$ first)

$$\lim_{\delta\to 0} \int_r^R \frac{(x+i\delta)^{-c}}{1+x+i\delta} dx = \int_r^R \frac{x^{-c}}{1+x} dx$$

(Consider $g(x,\delta) = \begin{cases} \frac{(x+i\delta)^{-c}}{1+x+i\delta} & \text{for } x\in[r,R],\ \delta\in(0,\frac{1}{100}] \\ \frac{x^{-c}}{1+x} & \text{for } x\in[r,R],\ \delta=0 \end{cases}$

Continuous on $[r,R]\times[0,\frac{1}{100}]$ $\Rightarrow$ uniform continuous

$\Rightarrow g(x,\delta) \xrightarrow{\delta\to 0} g(x,0)$ uniformly $\forall x\in[r,R]$)

Similarly $\lim_{\delta\to 0} \int_r^R \frac{(x-i\delta)^{-c}}{1+x+i\delta} dx = e^{-2\pi i c} \int_r^R \frac{x^{-c}}{1+x} dx$

$\Rightarrow (1-e^{-2\pi i c}) \boxed{?} = 2\pi i\, e^{-\pi i c}$ ($-$: from direction of γ_2)

$\Rightarrow \boxed{?} = \frac{2\pi i}{e^{\pi i c} - e^{-\pi i c}} = \frac{2\pi i}{2i\sin(\pi c)} = \frac{\pi}{\sin(\pi c)}$ #

<u>example 3</u> [Ahlfors, #5 on P.161]

$f(z)$: analytic and bounded on $D=\{z\in\mathbb{C} \mid |z|<1\}$

$\Rightarrow f(w) = \frac{1}{\pi}\iint_D \frac{f(z)}{(1-\bar{z}w)^2} dx\,dy \qquad \forall w\in D$

(double integral over the region)

sol'n : $z=re^{i\theta}$ polar coordinate

$$\frac{1}{\pi}\iint_D \frac{f(z)}{(1-\bar{z}w)^2} dx\,dy = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{f(re^{i\theta})}{(1-re^{-i\theta}w)^2} r\,dr\,d\theta$$

Fix $r\in(0,1)$, evaluate this integral

Similar to example 1, regard it as a line integral over $\partial B(0;r)$

$\zeta = re^{i\theta} \in \partial B(0,r)$

$d\zeta = ire^{i\theta}d\theta \Rightarrow d\theta = \frac{d\zeta}{i\zeta} \qquad re^{-i\theta} = \frac{r^2}{re^{i\theta}} = \frac{r^2}{\zeta}$

$$\frac{1}{\pi}\int_0^{2\pi} \frac{f(re^{i\theta})}{(1-re^{-i\theta}w)^2} d\theta = \frac{1}{\pi}\int_{\partial B(0,r)} \frac{f(\zeta)}{(1-\frac{r^2 w}{\zeta})^2}\frac{d\zeta}{i\zeta} = \frac{1}{\pi i}\int_{\partial B(0,r)} \frac{\zeta f(\zeta)}{(\zeta-r^2w)^2} d\zeta$$

$\frac{\zeta f(\zeta)}{(\zeta-r^2w)^2}$ has a pole of order 2 at $\zeta = r^2 w$ (enclosed by it: $|r^2w| < r^2 < r$)

$$\zeta f(\zeta) = (r^2w + (\zeta-r^2w))(f(r^2w) + f'(r^2w)(\zeta-r^2w) + (\zeta-r^2w)^2 g(\zeta))$$
$$= r^2w f(r^2w) + (f(r^2w) + r^2w f'(r^2w))(\zeta-r^2w) + (\zeta-r^2w)^2(\cdots)$$

$$\Rightarrow \operatorname{Res}_{\zeta=r^2w} \frac{\zeta f(\zeta)}{(\zeta-r^2w)^2} = f(r^2w) + r^2w f'(r^2w)$$

$$\Rightarrow \frac{1}{\pi}\int_0^{2\pi} \frac{f(re^{i\theta})}{(1-re^{-i\theta}w)^2}\, d\theta = \frac{2\pi i}{\pi i}\left(f(r^2w) + r^2 w f'(r^2 w)\right)$$

$$\frac{1}{\pi}\iint_D \frac{f(z)}{(1-\bar{z}w)^2}\, dx\, dy = \int_0^1 \left(f(r^2w) + r^2 w f'(r^2w)\right) 2r\, dr = r^2 f(r^2 w)\Big|_{r=0}^{1} = f(w) \quad \#$$

<u>rmk</u> This is called the Bergmann formula.

In general, $L^{2,h}(D)$ is a closed subspace in $L^2(D)$

$L^{2,h}(D)$ = square integrable holomorphic functions

$L^2(D)$ = square integrable functions

The (L^2)-orthogonal projection $L^2(D) \to L^{2,h}(D)$ is given by $g(z) \longmapsto \frac{1}{\pi}\iint_D \frac{g(z)}{(1-\bar{z}w)^2}\, dx\, dy$

(holomorphic)