IV. local structure: Taylor & Laurent series, open mapping theorem, maximum principle

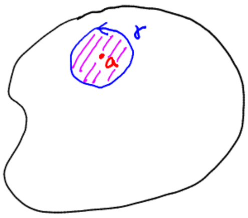
§1. uniform limit [Ahlfors, §1.1 of ch.5]

Before discussing the Taylor series, the limit of analytic functions is basically still analytic.

<u>thm</u> [Weierstrass] $\{f_n(z)\}, f(z)$: defined on Ω

If $f_n(z)$ is analytic and $f_n(z) \to f(z)$ uniformly on any compact subset of Ω, then $f(z)$ is analytic and $f_n'(z) \to f'(z)$ uniformly on any compact subset of Ω

pf:



It is a local issue. Choose $B(z_0;\rho)$ with $\overline{B(z_0;\rho)} \subset \Omega$, and let $\gamma = \partial B(z_0;\rho)$

For any $z \in B(z_0;\rho)$

$$f_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(\zeta)}{\zeta - z}\, d\zeta$$

$f_n(\zeta) \to f(\zeta)$ uniformly for $\zeta \in \gamma$

$\Rightarrow \dfrac{f_n(\zeta)}{\zeta - z} \to \dfrac{f(\zeta)}{\zeta - z}$ uniformly

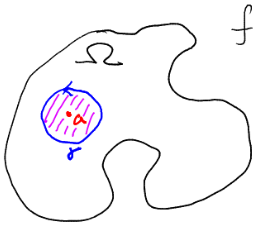
It follows that
$$f(z) = \lim_{n\to\infty} f_n(z) = \frac{1}{2\pi i}\int_\gamma \lim_{n\to\infty}\frac{f_n(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Hence, $f(z)$ is analytic on $B(z_0;\rho)$.

Similarly,
$$\left| f_n'(z) - f'(z) \right| \le \frac{1}{2\pi}\int_\gamma \frac{|f_n(z) - f(z)|}{|\zeta - z|^2}\,|d\zeta| \underset{\substack{\text{provided } z \in B(z_0;\frac{\rho}{2})}}{\le} \frac{1}{2\pi}\frac{4}{\rho^2}\int_\gamma |f_n(z) - f(z)|\,|d\zeta|$$

$\Rightarrow f_n'(z) \to f'(z)$ uniformly on $B(z_0;\frac{\rho}{2})$ ※

§2 Taylor series [Ahlfors, §1.2 of ch.5]



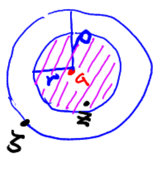
$f(z)$: analytic on Ω. For any $a \in \Omega$, choose $B(a;\rho)$ with $\overline{B(a;\rho)} \subset \Omega$; let $\gamma = \partial B(a;\rho)$

$$\Rightarrow f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + (z-a)^{n+1}\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}(\zeta - z)}\, d\zeta$$

Examine the remainder term for $z \in B(a;r)$ with $r < \rho$

$$\left| (z-a)^{n+1}\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}(\zeta-z)}\, d\zeta \right| \le \frac{r^{n+1}}{2\pi}\, \frac{\overset{= \sup|f(\zeta)|}{M}}{\rho^{n+1}\underbrace{(\rho - r)}}\, 2\pi\rho$$

$|\zeta - z| \ge |\zeta - a| - |z - a| \ge \rho - r$



That is to say, the remainder term $\xrightarrow{n\to\infty} 0$ uniformly on $B(a;r)$

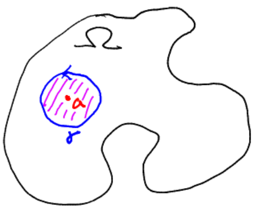
By the previous theorem $\left(\sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(z-a)^k \longrightarrow f(z) \right)$

$$f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots$$

on any $B(a;\rho) \subset \Omega$, and the convergence (as well as all the higher order derivatives) is uniformly on any compact subset of $B(a;\rho)$

In other words, we are allowed to do term by term differentiation (to any order) on $B(a;\rho)$

§3 Laurent series [Ahlfors, §1.3 of ch.5]



If $f(z)$ has a pole at $a \in \Omega$, what can we do?

recall : $f(z) = \frac{b_n}{(z-a)^n} + \cdots + \frac{b_1}{(z-a)} + g(z)$

$g(z)$ ↳ analytic on $B(a;\rho)$ ↙ admits the Taylor series

$$\sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!}(z-a)^k$$

Q what about essential singularities?

eg $\exp(\frac{1}{z})$ at $z=0$ $\rightsquigarrow$ $\exp(w) = 1 + w + \cdots + \frac{w^n}{n!} + \cdots$ $\forall w \in \mathbb{C}$ and uniformly for $|w| < R$

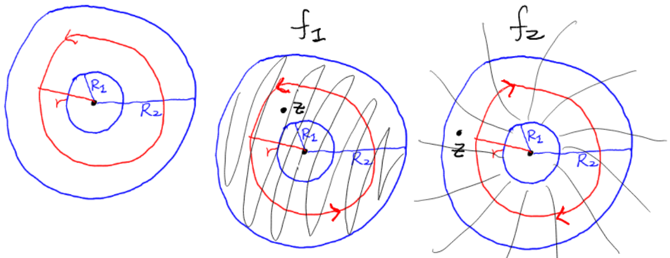
$\Rightarrow \exp(\frac{1}{z}) = 1 + z^{-1} + \cdots + \frac{z^{-n}}{n!} + \cdots$ $\forall z \neq 0$ and uniformly for $|z| > \frac{1}{R}$

Prop Suppose that $f(z)$ is an analytic function defined on the annulus $\{ z \in \mathbb{C} \mid R_1 < |z| < R_2 \}$. Then, $f(z)$ can be written as $f_1(z) + f_2(z)$

could be 0, ∞

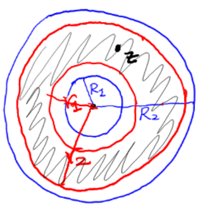
where $f_1(z)$ is defined and analytic on $\{|z| < R_2\}$

$f_2(z)$ is defined and analytic on $\{|z| > R_1\}$ and has $z=\infty$ to be a removable singularity



It follows that $f_1(z)$ admits a power series expansion in z on $\{|z| < R_2\}$, and $f_2(z)$ admits a power series expansion in $1/z$ on $\{|z| > R_1\}$ (converge uniformly on compact subsets)

Pf: Step 1) Given z, choose r_1 and r_2 such that $R_1 < r_1 < |z| < r_2 < R_2$



We can apply Cauchy's theorem for $\frac{f(w) - f(z)}{w - z}$ on $\Omega = \{ w \in \mathbb{C} \mid r_1 \le |z| \le r_2 \}$

↳ analytic on $\Omega \setminus \{z\}$ satisfying the limiting condition at z

$$\Rightarrow \int_{\partial\Omega} \frac{f(w) - f(z)}{w - z}\, dw = 0$$

$$\Rightarrow \quad f(z) = \frac{1}{2\pi i}\int_{|\zeta|=r_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta \;-\; \frac{1}{2\pi i}\int_{|\zeta|=r_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

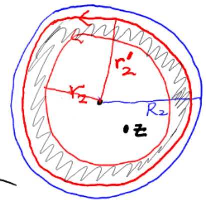
index $\begin{cases} |\zeta| = r_2 \\ |\zeta| = r_1 \end{cases}$ around z is $\begin{cases} 1 \\ 0 \end{cases}$ candidates for $f_1(z)$ and $f_2(z)$

Step 2) Given z, choose $r_2 \in (|z|, R_2)$

Define $f_1(z)$ to be $\displaystyle \frac{1}{2\pi i}\int_{|\zeta|=r_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta$

As explained last week, it is analytic on $|z| < r_2$

By Cauchy's theorem, it is independent of the choice of r_2.

[

$\frac{f(w)}{w-z}$ is analytic on $V = \{w \mid r_2 < |w| < r_2'\}$

$$\Rightarrow \int_{\partial V} \frac{f(w)}{w-z}\,dw = 0 \;\Rightarrow\; \int_{|\zeta|=r_2'} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \int_{|\zeta|=r_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$]

step 3) Given z, choose $r_1 \in (R_1, |z|)$

Define $f_2(z)$ to be $\displaystyle \frac{-1}{2\pi i}\int_{|\zeta|=r_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta$

By the same token, it is independent of the choice of r_1

$$f_2\left(\frac{1}{w}\right) = +\frac{1}{2\pi i}\int_{|\xi|=\frac{1}{r_1}} \frac{f(1/\xi)}{}\,d\xi \qquad \zeta = \frac{1}{\xi}$$

$$d\zeta = -\xi^{-2}\,d\xi$$

can be work out explicitly by $\zeta = r_1 e^{it}$, $\xi = \frac{1}{r_1}e^{-it}$ ...

$$= \underbrace{\frac{w}{2\pi i}}_{\text{vanishes at } w=0 \;\Leftrightarrow\; z=\infty} \underbrace{\int_{|\xi|=\frac{1}{r_1}} \frac{f(1/\xi)}{(\xi - w)\,\xi}\,d\xi}_{\text{analytic on } |w| < \frac{1}{r_1}} \qquad \left(\frac{1}{\xi} - \frac{1}{w}\right)(-\xi^2) = \frac{\xi(\xi - w)}{w}$$

#

rmk. i)
$$f(z) = f_1(z) + f_2(z) = \sum_{n=0}^{\infty} A_n z^n + \sum_{n=1}^{\infty} A_{-n} z^{-n}$$

By the theory of power series (radius of convergence ...)

$$\sum_{n=0}^{\infty} \frac{A_n}{n+1} z^{n+1} + \sum_{n=2}^{\infty} \frac{A_{-n}}{1-n} z^{1-n}$$ defines an analytic function on the annulus.

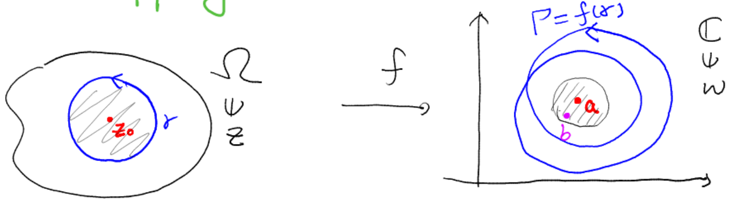
$$\frac{d}{dz}\Big(\quad \sum_{n=0}^{\infty} A_n z^n + \sum_{n=2}^{\infty} A_{-n} z^{-n}$$: $A_{-1} z^{-1}$ is missing

anti-derivative is $\log z$

ii) By using the above discussion and Abel's power series theorem, it is not hard to prove the uniqueness of the Laurent series.

iii) Therefore, one can also use the Laurent series to characterize the nature of an isolated singularity.

§4 open mapping theorem [Ahlfors, §3.3 of ch.5]



If $f(z)$ is not a constant function,

$f(z) - a = (z-z_0)^n g(z)$, $n \in \mathbb{N}$

$g(z)$: defined and analytic on a nbd of a. Then, by removable of singularity, $\frac{f(z)-a}{(z-z_0)^n}$ is analytic on Ω

$\Rightarrow f(z) = a + g(z_0)(z-z_0)^n + $ (higher order terms)

Intuitively, the local behavior of $f(z)$ shall be similar to $z \mapsto z^n$

(See [Ahlfors, P.133] for the precise construction, and it can be used to show the open mapping theorem)

Moreover, $f(z)$ has to map open sets to open sets.

Let us present a more complex analytical argument:

<u>step 1</u> For ε: sufficiently small, $f(z) \neq a$ for $|z-z_0| \leq \varepsilon$ and $z \neq z_0$ $(g(z) \neq 0)$

Let $\gamma = \partial B(z_0; \varepsilon)$. $\Gamma = f(\gamma)$ is a closed curve in $\mathbb{C}$

Since $a \notin \Gamma$, $\exists\, \delta > 0$ such that $B(a;\delta)$ does not intersect Γ

<u>step 2</u> For any b in $B(a;\delta)$, consider the index of Γ around b:

$$\int_\Gamma \frac{dw}{w-b} = \int_\Gamma \frac{dw}{w-a} = \int_{f(\gamma)} \frac{d f(z)}{f(z)-a} = \int_\gamma \frac{f'(z)}{f(z)-a}\,dz = (2\pi i)\, n$$

$$f(z) - a = (z-z_0)^n g(z)$$

$$f'(z) = n(z-z_0)^{n-1} g(z) + (z-z_0)^n g'(z)$$

$$\Rightarrow \frac{f'(z)}{f(z)-a} = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}$$

analytic on $\overline{B(z_0;\varepsilon)}$ $\Rightarrow \int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$

In fact, $(\log g(z))'$ is well-defined on $B(z_0;\varepsilon)$

<u>step 3</u> $$\int_\Gamma \frac{dw}{w-b} = \int_\gamma \frac{f'(z)}{f(z)-b}\,dz = (2\pi i)\, n$$

- Remember that $f(z) \neq b$ on γ. If $f(z) \neq b$ for any $z \in B(z_0;\varepsilon)$, $\frac{f'(z)}{f(z)-b}$ is analytic on $\overline{B(z_0;\varepsilon)}$ $\xrightarrow{\text{Cauchy}} \int_\gamma \frac{f'(z)}{f(z)-b}\,dz = 0$ $\rightarrow\!\leftarrow$
- Since the zeros of $f(z)-b$ cannot have accumulation point in $B(z_0;\varepsilon)$ (also $f(z) \neq b$ on $\partial B(z_0;\varepsilon)$) $\Rightarrow$ the zeros of $f(z)-b$ in $B(z_0;\varepsilon)$: finite number of points, $\{z_1, \ldots, z_m\}$ repeat to the multiplicity.
- By the local behavior of zeros and the removable of singularity lemma

$$h(z) = \frac{f(z)-b}{(z-z_1)\cdots(z-z_m)}$$ is a nowhere zero analytic function on $\overline{B(z_0;\varepsilon)}$

• $$\frac{f'(z)}{f(z)-b} = \frac{1}{z-z_1} + \cdots + \frac{1}{z-z_m} + \frac{h'(z)}{h(z)} \Rightarrow \int_\gamma \frac{f'(z)}{f(z)-b}\,dz = (2\pi i)\, m$$

$$\Rightarrow m = n$$

※

Upshot

<u>thm</u> $f(z)$ = analytic at z_0, $f(z_0) = a$

The order of $f(z) = a$ has order $n < \infty$ at z_0

Then, for sufficiently small $\varepsilon > 0$, $\exists\, \delta > 0$ such that any $b \in B(a;\delta)$, $f(z) = b$ has exactly n roots in $B(z_0;\varepsilon)$

(In particular, $B(a;\delta) \subset f(B(z_0;\varepsilon))$)

§5 maximum principle [Ahlfors, §3.4 of ch.5]

<u>thm</u> (maximum principle) $f(z)$: analytic on Ω ← open and connected

If $f(z)$ is non-constant, $|f(z)|$ has no maximum in Ω

pf: It follows directly from the open mapping theorem. #

(One can also prove it by using the Cauchy integral formula, see [Ahlfors, P.134 ~ 135])

<u>rmk</u> Equivalently, $f(z)$: analytic on $\overline{\Omega}$ ⇒ maximum of $f(z)$ happens on $\partial\Omega$

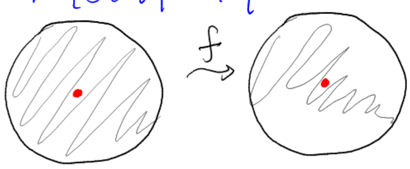
It can be used to studied the automorphisms of regions in $\mathbb{C}$.

Here is the key tool to study Aut (Disk)

<u>thm</u> (Schwarz lemma) $f(z)$: analytic for $|z| < 1$ with $|f(z)| \le 1$, $f(0) = 0$. Then $|f(z)| \le |z|$ and $|f'(0)| \le 1$

Moreover, if (one of) the equality holds, then $f(z) = cz$ for some constant c with $|c| = 1$

$\left\{ \begin{array}{l} |f'(0)| = 1 \text{ or} \\ |f(z)| = |z| \text{ for some } z \neq 0 \end{array} \right.$



pf: Consider $g(z) = \dfrac{f(z)}{z}$

$\lim_{z \to 0} z\, g(z) = 0 \Rightarrow g(z)$ has a removable singularity at $z = 0$

Since $\lim_{z \to 0} g(z) = f'(0)$, $g(0)$ shall be defined to be $f'(0)$

Consider $g(z)$ on $B(0;r)$ for $r < 1$. Since $|g(z)| \le \frac{1}{r}$ on

Since $|g(z)| \le \frac{1}{r}$ on $\partial B(0;r)$, $|g(z)| \le \frac{1}{r}$ on $B(0;r)$

It is true for any $r < 1$. Thus $|g(z)| \le 1$ on $B(0;1)$

If the equality holds, $g(z)$ must be a constant #

Cor $f(z)$ analytic on $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$, $|f(z)| < 1 \quad \forall z \in D$

If $f(z)$ is bijective and $f(0) = 0$, then $f(z) = cz$

for some constant c with $|c| = 1$

pf: It can be shown that $h = f^{-1}$ is also analytic

$h(0) = 0$ By the Schwarz lemma. $|f'(0)| \le 1$ & $|h'(0)| \le 1$

But $h \circ f = z \Rightarrow h'(0) f'(0) = 1$

$\Rightarrow |f'(0)| = 1 = |h'(0)|$: the equality case in the Schwarz lemma #