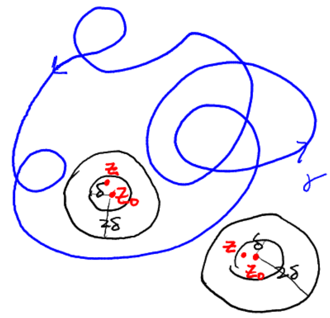
III. local structure : zeros and poles , isolated singularities

§1 (higher order) derivative



lemma γ : closed curve on $\mathbb{C}$, $\varphi(\zeta)$: continuous on γ

Then, $F_n(z) = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n}\, d\zeta \quad (n \in \mathbb{N})$ is analytic on each region of $\mathbb{C} \setminus \{\gamma\}$, and $F_n'(z) = n F_{n+1}(z)$

pf: For any $z_0 \in \mathbb{C} \setminus \{\gamma\}$, $\exists \delta > 0$ such that $B(z_0, 2\delta) \subset \mathbb{C} \setminus \{\gamma\}$

Consider $z \in B(z_0, \delta)$

(step 1 : continuity of $F_1(z)$)

$$F_1(z) - F_1(z_0) = \int_\gamma \left(\frac{\varphi(\zeta)}{\zeta - z} - \frac{\varphi(\zeta)}{\zeta - z_0} \right) d\zeta = (z - z_0) \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)(\zeta - z_0)}\, d\zeta$$

Since $\zeta \in \{\gamma\}$, $|\zeta - z| > \delta$, $|\zeta - z_0| > 2\delta$

$$\Rightarrow \left| F_1(z) - F_1(z_0) \right| < |z - z_0| \frac{1}{2\delta^2} \int_\gamma |\varphi(\zeta)|\, |d\zeta|$$

(step 2 : $F_1'(z) = F_2(z)$) $\quad \dfrac{F_1(z) - F_1(z_0)}{z - z_0} = \displaystyle\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)(\zeta - z_0)}\, d\zeta$: $F_1(z)$ of $\dfrac{\varphi(\zeta)}{\zeta - z_0}$

$$\Big\downarrow \text{ by step 1, as } z \to z_0$$

$$F_2(z_0) = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^2}\, d\zeta$$

skip this part in class

(step 3 : $F_n'(z) = n F_{n+1}(z)$) Prove it by induction; assume $F_{n-1}'(z) = (n-1) F_n(z)$

$$F_n(z) - F_n(z_0) = \int_\gamma \left(\frac{\varphi(\zeta)}{(\zeta - z)^n} - \frac{\varphi(\zeta)}{(\zeta - z_0)^n} \right) d\zeta$$

$$= \int_\gamma \left[\left(\frac{1}{(\zeta - z)^n} - \frac{1}{(\zeta - z)^{n-1}(\zeta - z_0)} \right) + \left(\frac{1}{(\zeta - z)^{n-1}(\zeta - z_0)} - \frac{1}{(\zeta - z_0)^n} \right) \right] \varphi(\zeta)\, d\zeta$$

$$= (z - z_0) \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n (\zeta - z_0)}\, d\zeta + \left(F_{n-1}\left(z ; \frac{\varphi(\zeta)}{\zeta - z_0}\right) - F_{n-1}\left(z_0 ; \frac{\varphi(\zeta)}{\zeta - z_0}\right) \right)$$

By the same token, $F_n(z)$ is continuous.

Then,
$$\frac{F_n(z) - F_n(z_0)}{z - z_0} = F_n\left(z ; \frac{\varphi(\zeta)}{\zeta - z_0}\right) + \frac{F_{n-1}\left(z ; \frac{\varphi(\zeta)}{\zeta - z_0}\right) - F_{n-1}\left(z_0 ; \frac{\varphi(\zeta)}{\zeta - z_0}\right)}{z - z_0}$$

by the continuity of F_n $\Big\downarrow$ as $z \to z_0$ $\qquad$ by induction hypothesis $\Big\downarrow$ as $z \to z_0$

$$n F_{n+1}(z_0) = F_n\left(z_0, \frac{\varphi(\zeta)}{\zeta - z_0}\right) + (n-1) F_n\left(z_0, \frac{\varphi(\zeta)}{\zeta - z_0}\right)$$

※

APPLICATIONS

thm (Morera) $f \in \mathcal{C}^0(\Omega)$ if $\int_\gamma f\, dz = 0$ for any closed curve γ in Ω, then f is analytic.

[pf: We have shown that $f(z) = \mathcal{U}'(z)$, where $\mathcal{U}(z)$ is analytic

The above lemma and the Cauchy integral formula imply that the derivative of an analytic function is still analytic #]

thm (Liouville) $f(z)$: entire function (defined and analytic on $\mathbb{C}$)
If $f(z)$ is bounded, i.e. $|f(z)| \le M \quad \forall z \in \mathbb{C}$
then $f(z) \equiv$ constant

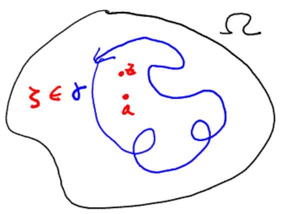
pf: It suffices to show that $f'(z) \equiv 0$

$$f'(z) = \frac{1}{2\pi i} \int_{\partial B(z;\rho)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \Rightarrow |f'(z)| \le \frac{1}{2\pi} \frac{M \cdot 2\pi\rho}{\rho^2} = \frac{M}{\rho} \xrightarrow{\rho\to\infty} 0 \quad \#$$

thm (fundamental theorem of algebra) Any nontrivial polynomial admits a root ← leave its proof as a homework

§2 Taylor's theorem

Since an analytic function is differentiable to any order, the Taylor theorem holds. In fact, it is rather easy to construct the remainder term by using the Cauchy integral formula.



$f(z)$ = analytic on $\Omega \ni a$

choose a closed curve γ in Ω with $n(\gamma, a) = 1$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$ for any z in the same component (of $\mathbb{C} \setminus \{\gamma\}$) as a

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} + \frac{z-a}{\zeta - a}\frac{1}{\zeta - z}$$

repeat

$$\Rightarrow f(z) = f(a) + (z-a)\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)(\zeta - z)}\, d\zeta$$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} + \frac{z-a}{\zeta - a}\left(\frac{1}{\zeta - a} + \frac{z-a}{\zeta - a}\frac{1}{\zeta - z}\right)$$

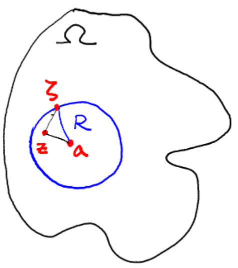
$$= \frac{1}{\zeta - a} + \frac{z-a}{(\zeta - a)^2} + \frac{(z-a)^2}{(\zeta - a)^2}\frac{1}{\zeta - z}$$

$$= \cdots = \frac{1}{\zeta - a} + \cdots + \frac{(z-a)^{n-1}}{(\zeta - a)^n} + \frac{(z-a)^n}{(\zeta - a)^n}\frac{1}{\zeta - z}$$

$$\Rightarrow f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + (z-a)^n \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - a)^n(\zeta - z)}\, d\zeta$$

APPLICATIONS: local structure of zeros of an analytic function

Cor Ω : open and connected, $f(z)$: analytic on Ω
If $f^{(n)}(a) = 0 \quad \forall n \ge 0$, then $f(z) \equiv 0$ on Ω



pf: claim ∃ open disk at a on where $f(z) \equiv 0$

$$f(z) = (z-a)^n \frac{1}{2\pi i}\int_{\partial B(a;R)} \frac{f(\zeta)}{(\zeta - a)^n(\zeta - z)}\, d\zeta \quad \forall n \ge 1$$

$$\Rightarrow |f(z)| \le |z-a|^n \frac{1}{2\pi}\frac{2\pi R\, M}{R^n (R - |z-a|)}$$

$$\begin{cases} M = \sup_{\partial B(a;R)} |f(\zeta)| \\ |\zeta - z| \ge |\zeta - a| - |z-a| = R - |z-a| \end{cases}$$

$\downarrow$ as $n \to \infty$ (for any fixed z with $|z-a| < R$)
0

$$\left| \begin{array}{l} E_1 = \{ z \in \Omega \mid f^{(n)}(z) = 0 \quad \forall\, n \geq 0 \} : \text{open (by the claim)} \\ E_2 = \{ z \in \Omega \mid f^{(n)}(z) \neq 0 \ \text{ for some } n \geq 0 \} : \text{open} \ \Rightarrow E_2 = \phi_{\#} \end{array} \right|$$

If $f^{(n)}(a) \neq 0$ but $f^{(k)}(a) = 0 \quad \forall\, k \in \{0, 1, \cdots, n-1\}$

$$\Rightarrow f(z) = (z-a)^n \left(\underbrace{\frac{f^{(n)}(a)}{n!} + (z-a) \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta-a)^{(n+1)}(\zeta - z)} d\zeta}_{\text{nonzero on some nbd of } a} \right)$$

Equivalently,

<u>Cor</u> Ω : open and connected, $f(z)$ and $g(z)$: analytic on Ω

If $f = g$ on a set with accumulation point in Ω

then $f \equiv g$

<u>Note</u> that
$$\lim_{z \to a} |z-a|^{\alpha} |f(z)| = \begin{cases} 0 & \alpha > -n \\ \infty & \alpha < -n \\ \left(\dfrac{|f^{(n)}(a)|}{n!} \right. & \left. \alpha = -n \right) \end{cases} \qquad -n : \text{critical exponent}$$

$\alpha + n = 0$

Usually, we say a is a zero of order n.

§3 poles

We have learned that an analytic function is determined by its value nearby

Q i) "maximal domain" of an analytic function ?

ii) possible isolated singularities, e.g. $\frac{1}{z}$, $\frac{1}{z^n}$ at $z=0$

closely related to the theory of zeros

<u>lemma</u> [removable singularity] $g(z)$ = analytic on $\Omega \setminus \{a\}$, Then

$\lim_{z \to a} (z-a)\, g(z) = 0$ if and only if $g(z)$ admits a (unique) analytic extension on Ω

pf: $\Leftarrow$) Straightforward

$\Rightarrow$) Choose δ : sufficiently small such that $\overline{B(a;\delta)} \subset \Omega$

Let γ be its boundary curve.

For any $w \in B(a;\delta) \setminus \{a\}$, $\dfrac{g(z) - g(w)}{z - w}$ is analytic except at a and w

But
$$\lim_{z \to w} (z-w) \frac{g(z) - g(w)}{z - w} = 0$$
$$\lim_{z \to a} (z-a) \frac{g(z) - g(w)}{\underbrace{z-w}_{\to a-w}} = 0$$

$$\Rightarrow \underbrace{g(w)}_{\text{analytic on } \Omega \setminus \{a\}} = \frac{1}{2\pi i} \int_\gamma \underbrace{\frac{g(z)}{z-w} dw}_{\text{analytic on } B(a;\delta)} \qquad \#$$

Now, let us examine the "blow-up" case.

That is to say, $f(z)$ is analytic on $\Omega \setminus \{a\}$

and $\lim_{z \to a} f(z) = \infty$

discussion: $|f(z)| \gg 1$ on a small neighborhood B of $\{a\}$

(More precisely, $B \setminus \{a\}$)

$\Rightarrow \frac{1}{f(z)}$ is defined and analytic on $B \setminus \{a\}$

Since $\lim_{z \to a} \frac{1}{f(z)} = 0 \Rightarrow \lim_{z \to a} \frac{z-a}{f(z)} = 0$, the removable of singularity lemma

asserts that $\frac{1}{f(z)}$ admits an analytic extension to $z=a$

$\frac{1}{f(a)}$ must be zero.

But a must be an isolated zero of $\frac{1}{f(z)}$

Therefore. $\frac{1}{f(z)} = (z-a)^n g(z)$ for some $n \in \mathbb{N}$

and $g(z)$: analytic on a neighborhood of a with $g(a) \neq 0$

$$\Rightarrow f(z) = \frac{1}{(z-a)^n} h(z) \qquad h(a) \neq 0$$

By the Taylor's theorem (on $h(z)$)

$$f(z) = \underbrace{\frac{b_n}{(z-a)^n} + \frac{b_{n-1}}{(z-a)^{n-1}} + \cdots + \frac{b_1}{z-a}}_{\text{singular part of } f(z) \text{ at } a} + h_{n+1}(z)$$

nonzero

analytic on a neighborhood of a

a is called a pole of order n

Note that

$$\lim_{z \to a} |z-a|^{\alpha} |f(z)| = \begin{cases} 0 & \alpha > n \\ \infty & \alpha < n \\ (|b_n| & \alpha = n) \end{cases}$$

$\alpha - n = 0$

§4 Isolated singularities

Based on the above discussions.

If $f(z)$: analytic on $\Omega \setminus \{a\}$. we can try to use

$\lim_{z \to a} |z-a|^{\alpha} |f(z)|$ to detect the behavior of $f(z)$ at a

Next time, we will see that it basically belongs to the above two cases, with one exception situation

$\left(\text{e.g. } f(z) = \exp\left(\frac{1}{z}\right) \text{ at } z=0\right)$

① $\lim_{z\to a} |z-a|^{\alpha}|f(z)| = 0$

② $\lim_{z\to a} |z-a|^{\alpha}|f(z)| = \infty$ $\qquad \alpha \in \mathbb{R}$

This page : potentially next week

- If ① holds for some α,
  it is also true for $n \geq \alpha$, $n \in \mathbb{Z}$
  $\Rightarrow \lim_{z\to a} |z-a|^{n}|f(z)| = 0$
  $\Rightarrow (z-a)^{n-1} f(z)$ is analytic on a neighborhood of a
  $\Rightarrow f(z)$ has at worst a pole at a (i.e. either analytic or a pole)

- If ② holds for some α,
  it is also true for $n < \alpha$, $n \in \mathbb{Z}$
  $\Rightarrow \lim_{z\to a} |z-a|^{n}|f(z)| = \infty$
  $\Rightarrow (z-a)^{n} f(z)$ has a pole at $z=a$
  $\Rightarrow f(z)$ has at worst a pole at a

To sum up, there are three possibilities

$\boxed{\text{i}}$ condition ① holds for all α $\Rightarrow f \equiv 0$

$\boxed{\text{ii}}$ $\exists\, n \in \mathbb{Z}$ such that ① for $\alpha > n$
② for $\alpha < n$

and $n < 0 \iff a$ is a zero
$n > 0 \iff a$ is a pole

$\boxed{\text{iii}}$ Neither ① nor ② holds for any $\alpha \in \mathbb{R}$
In this case, a is called an <u>essential singularity</u> (ess. sing. for short)

The local behavior near an essential singularity is weird.
(We know it is not going to ∞)

<u>prop</u> If a is an <u>essential</u> singularity of $f(z)$
then $\overline{f(B(a;\rho)\setminus\{a\})} = \mathbb{C}$ for any $\rho > 0$

pf: If NOT, $\exists\, B(A;\delta)$ in the complement of $\overline{f(B(a;\rho)\setminus\{a\})}$ (for any small $\rho>0$)
$\Rightarrow |f(z)-A| > \delta$ for any $z \in B(a;\rho)\setminus\{a\}$
$\Rightarrow \lim_{z\to a} |z-a|^{-1}|f(z)-A| = \infty$
$\Rightarrow a$ is NOT an essential singularity of $f(z)-A$ $\rightarrow\leftarrow$

<u>rmk</u> Picard's Great theorem
In fact $f(B(a;\rho)\setminus\{a\}) = \mathbb{C}$ or $\mathbb{C}\setminus\{q\}$ for some q
(i.e. f can miss at most one value)