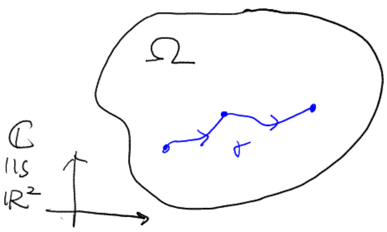
II. Cauchy integral formula

§1. line integrals



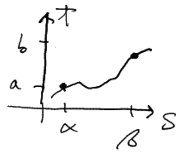
$\xrightarrow{f} \mathbb{C}$

$f : \underset{\text{open}}{\Omega \subset \mathbb{C}} \to \mathbb{C}$ a continuous function

$\gamma : t \to z(t)$ $\quad t \in [a,b]$ a piecewise smooth arc in Ω

Set $\int_\gamma f dz = \int_a^b f(z(t))\, z'(t)\, dt$ (It is understood that the integral is performed on each interval where $z(t)$ is smooth)

- According to the change of variable formula, $t = t(s)$



$$\int_a^b f(z(t))\, z'(t)\, dt = \int_\alpha^\beta f(z(t(s)))\, z'(t(s))\, t'(s)\, ds$$

Namely, $\int_\gamma f dz$ only depends on γ, but not how it is parametrized.

- With this understood, we usually like to think $\int_\gamma f dz$ as a function (or functional) of arcs. If f is analytic, it turns out that this function behaves nicely, and can be used to say more about analytic functions.

- $-\gamma$: opposite direction of γ. It is not hard to show that
$$\int_{-\gamma} f\, dz = -\int_\gamma f dz$$

- notations : $\quad f(z) = u(z) + i v(z) \qquad z(t) = x(t) + i y(t)$
$$\int_\gamma f dz = \int_a^b (u+iv)(x'+iy')\, dt = \int_a^b (u+iv)\, dx + (iu - v)\, dy$$

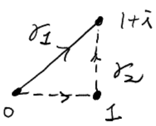
In general, we can consider the line integral of $\underline{p\, dx + q\, dy}$

$f = u + iv \rightsquigarrow \quad p = f \quad , q = i f$

<u>rmk</u> (i) $p dx + q dy$ is called a "differential 1-form" on $\mathbb{C}$. In differential geometry, what can be integrated over a "k-dimensional object" is a "k-form".

(ii) The theories work for a more general classes of arcs, "rectifiable" arcs. They are arcs whose arc length is well-defined, and over which we can consider line integrals. You can find more in [Nevanlinna, §8.1]

<u>e.g.</u>



$f(z) = z \qquad g(z) = \frac{z+\bar{z}}{2} = x$

$$\int_{\gamma_1} f\, dz = \int_{\gamma_1} (x dx - y dy) + i (y dx + x dy) \overset{\substack{x=t \\ y=t}}{=} \int_0^1 2 i t\, dt = i$$

$$\int_{\gamma_2} f\, dz = \int_0^1 0\, dt + i\, dt = i$$

same

$$\int_{\gamma_1} g\, dz = \int_{\gamma_1} x dx + i x dy = \frac{1+i}{2} \qquad \int_{\gamma_2} g\, dz = \frac{1}{2} + i$$

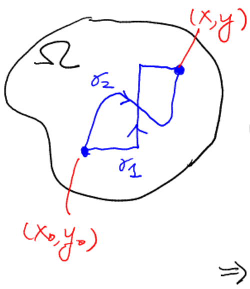
different

§2 exact differential

The space of all arcs in Ω is quite big, (actually, it is an infinite dimensional space). There are some special 1-forms whose line integral is better behaved.

<u>Thm 1</u> $p, q \in C^0(U)$. $\int_\gamma p\,dx + q\,dy$ depends only on the endpoints of γ if and only if $p\,dx + q\,dy$ is an exact differential

i.e. $\exists\, U \in C^1(\Omega)$ such that $p = \frac{\partial U}{\partial x}$ and $q = \frac{\partial U}{\partial y}$



$$\int_{\gamma_1} p\,dx + q\,dy = \int_{\gamma_2} p\,dx + q\,dy$$

pf: $\Leftarrow$) By the fundamental theorem of calculus

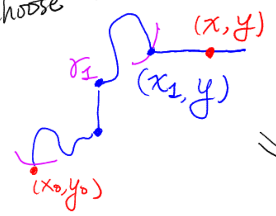
$$\int_\gamma \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = \int_a^b \left(\frac{\partial U}{\partial x}x' + \frac{\partial U}{\partial y}y'\right)dy = U(\gamma(b)) - U(\gamma(a))$$

$\Rightarrow$) Fix $(x_0,y_0) \in \Omega$. define $U(x,y) = \int_\gamma p\,dx + q\,dy$

where γ is an arc connecting (x_0,y_0) & (x,y)

(By assumption, it is independent of the choice of γ)

choose this arc



$\frac{\partial U}{\partial x} \overset{?}{=} p$ $\qquad U(x,y) = \int_{\gamma_1} p\,dx + q\,dy + \int_0^{x-x_1} p(x_1+t, y)\,dt$

$\Rightarrow \frac{\partial U}{\partial x} = p(x,y)$ (same argument for $\frac{\partial U}{\partial y}$) #

- Given $f \in C^0(\Omega)$, when does $f\,dz$ exact?

  If so, $\exists\, U$ such that $\frac{\partial U}{\partial x} = f$, $\frac{\partial U}{\partial y} = if$

  $\Rightarrow \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)U = 0 \Rightarrow U$ is analytic

  From the discussion last week $\Rightarrow U' = f$

Namely, $\int_\gamma f\,dz$ is "independent of path" if and only if $f = U'$ for some analytic function U.

Q i) $\int_\gamma f\,dz$: independent of path $\overset{?}{\Rightarrow}$ f: analytic

ii) f analytic $\overset{?}{\Rightarrow}$ f': analytic

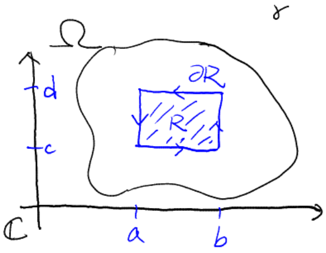
e.g. $U(z) = \frac{(z-z_0)^{n+1}}{n+1}$ $\quad U' = (z-z_0)^n \quad n \geq 0$

$\Rightarrow \int_\gamma (z-z_0)^n dz$ is independent of path

§3 Cauchy's theorem ($\square$)

$\int_\gamma p\,dx + q\,dy$: independent of path $\Leftrightarrow$ $\int_\gamma p\,dx + q\,dy = 0$ $\forall$ closed curve γ

closed: starting point = ending point

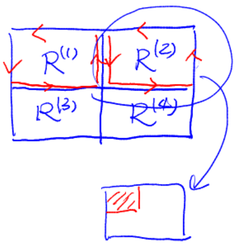


<u>thm</u> If $f(z)$ is analytic on R

($\exists\, \Omega \supset R$, f: analytic on Ω)

then $\int_{\partial R} f(z)\,dz = 0$

pf: (Quick way: invoke Stokes' (Green's) theorem provided $f \in C^1$)
Ahlfors presented a direct argument, due to Goursat, which doesn't require f to be C^1.



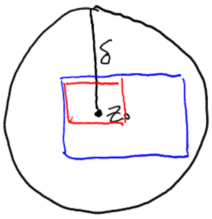
1° $\int_{\partial R} f\,dz = \int_{\partial R^{(1)}} f\,dz + \cdots + \int_{\partial R^{(2)}} f\,dz$

$\Rightarrow \exists$ one of them : R_1 such that $\left|\int_{\partial R_1} f\,dz\right| \geq \frac{1}{4}\left|\int_{\partial R} f\,dz\right|$

$\cdots \Rightarrow \exists\ R \supset R_1 \supset R_2 \supset \cdots R_n \supset \cdots$, each R_k is $\frac{1}{4}$ of R_{k-1}

such that $\left|\int_{\partial R_n} f\,dz\right| \geq \frac{1}{4^n}\left|\int_{\partial R} f\,dz\right|$

nested compact set in $\mathbb{R}^2$, diameter $\to 0$ $\Rightarrow \bigcap_{n=0}^{\infty} R_n = \{z_0\} \in R$



2° Study $\int_{\partial R_n} f\,dz$ for $n \gg 1$.

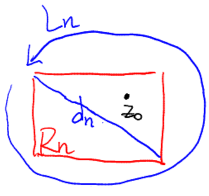
Given $\varepsilon > 0$ $\exists\, \delta > 0$ such that

$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0|$ for $|z - z_0| < \delta$

Then, $\exists N$ such that $R_n \subset B(z_0; \delta)$ for $n \geq N$

Since $\int_{\partial R_n} f(z_0)\,dz = 0 = \int_{\partial R_n} f'(z_0)(z - z_0)\,dz$

$$\int_{\partial R_n} f(z)\,dz = \int_{\partial R_n} \left(f(z) - f(z_0) - f'(z_0)(z - z_0)\right) dz$$



$$\Rightarrow \left|\int_{\partial R_n} f(z)\,dz\right| \leq \varepsilon \int_{\partial R_n} \underbrace{|z - z_0|}_{\leq d_n} \underbrace{|dz|}_{\leq L_n}$$

$dz = (x'(t) + i\,y'(t))\,dt$
$|dz| = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

$= \varepsilon (2^{-n} d)(2^{-n} L) = 4^{-n} \varepsilon\, d L$

diameter & perimeter of R

3° $\left|\int_{\partial R} f\,dz\right| \leq 4^n 4^{-n} \varepsilon\, d L \Rightarrow \int_{\partial R} f\,dz = 0$ #

any given

The result can be generalized to the boundary of other domains

e.g.

$\int_{\partial \Delta} f\,dz = 0$: $\Delta \subset$ rectangular domain

By the Cauchy theorem, $\int_\gamma f\,dz$ is independent of "rectangular" path.

$\Rightarrow$ It is enough to define U, i.e. $f\,dz$ is exact

$\Rightarrow \int_\gamma f\,dz$ is independent of path.

- What happened if f is NOT defined on R or Δ?

e.g. $f(z) = \frac{1}{z - a}$ Δ: disk of radius ρ at a.

$\int_{\partial \Delta} f\,dz = ?$ $\partial \Delta = \{ a + \rho e^{it} \mid 0 \leq t \leq 2\pi \}$

$$= \int_0^{2\pi} \frac{1}{\rho e^{it}}\, d(\rho e^{it}) = \int_0^{2\pi} \frac{i \rho e^{it}}{\rho e^{it}}\, dt = 2\pi i \neq 0$$

What about $\frac{1}{(z-a)^k}$ for $k \geq 2$?

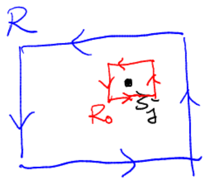
Based on this example, we have a mild generalization of Cauchy theorem. (Later on, we will realize that it is a "fake generalization".)

<u>thm</u> $R' =$ rectangle $R \setminus \{ \zeta_j :$ finite number of interior points$\}$

If $f(z)$ is analytic on R' and $\lim_{z \to \zeta_j} (z-\zeta_j) f(z) = 0$

then $\int_{\partial R} f(z)\, dz = 0$

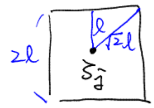
pf: It suffices to prove that $\left| \int_{\partial R_0} f(z)\, dz \right|$ is small for any "small" square centered at ζ_j.



$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(z)| < \frac{\varepsilon}{|z-\zeta_j|}$ for $|z-\zeta_j| < \delta$

If $R_0 \subset B(\zeta_j; \delta)$, $\left| \int_{\partial R_0} f(z) dz \right| < \varepsilon \int_{\partial R_0} \frac{|dz|}{|z-\zeta_j|}$

$< 8 \varepsilon$ #



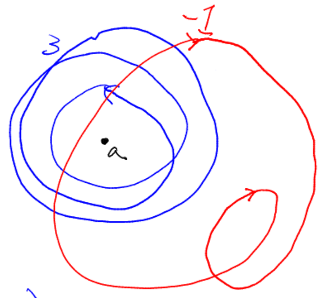
<u>rmk</u> The same conclusion holds for other regions.

[<u>Next time</u>, we will discuss what can be said if $\lim_{z \to \zeta_j} (z-\zeta_j) f(z)$ exists, but nonzero.]

§4 Index

<u>recall</u> $\Delta = B(a;\rho)$ $\qquad \displaystyle\int_{\partial\Delta} \frac{dz}{z-a} = 2\pi i$

γ = a closed curve does NOT pass through a



<u>claim</u> $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \in \mathbb{Z}$, roughly speaking, it counts how many times γ winds around a

$\left(\dfrac{dz}{z-a} \text{ "=" } d\log(z-a) = d\log|z-a| + i\, d\arg(z-a) \right)$

pf: $\gamma = \{ z(t) \mid 0 \le t \le 1 \}$ $\quad z(t)$: continuous and piecewise smooth

$\displaystyle\int_\gamma \frac{dz}{z-a} = \int_0^1 \frac{z'(t)}{z(t)-a}\, dt$ $\qquad \sim \log(z(t)-a) \Rightarrow e^{h(t)} \sim z(t)-a \Rightarrow (z(t)-a)e^{-h(t)}$ $\sim$constant

Consider $\displaystyle h(t) = \int_0^t \frac{z'(s)}{z(s)-a}\, ds$: continuous and piecewise smooth on $[0,1]$

$\Rightarrow h'(t) = \dfrac{z'(t)}{z(t)-a}$ (except a finite number of points)

$\Rightarrow (z(t)-a)h'(t) - z'(t) = 0 \Rightarrow \left((z(t)-a)e^{-h(t)}\right)' = 0$

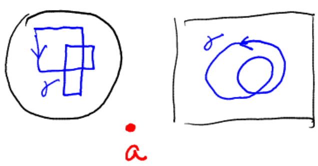
$\Rightarrow (z(t)-a)e^{-h(t)} = z(0)-a$

$\Rightarrow e^{-h(1)} = 1 \qquad \Rightarrow h(1) \in 2\pi i\,\mathbb{Z}$ #

<u>defn</u> The index of γ (not passing through a) is defined to be

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

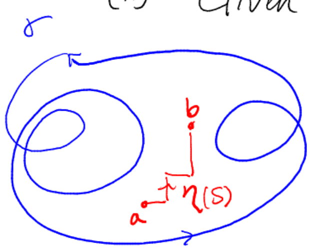
<u>properties</u> (i) if $\gamma \subset \Delta$ (or R)



$\Rightarrow n(\gamma, a) = 0$ for any $a \notin \Delta$ (or R)

[pf: Since $\frac{1}{z-a}$ is analytic on Δ or R, ...]

(ii) Given a closed curve γ, $\mathbb{C} \setminus \gamma = \coprod$ open set



For a, b in the same component

$$\int_\gamma \frac{1}{z-a}\, dz = \int_\gamma \frac{1}{z-b}\, dz$$

For a in the <u>unbounded</u> component $\qquad \hat{\mathbb{C}} \setminus \gamma = \coprod$ open set

$$\int_\gamma \frac{1}{z-a}\, dz = 0$$

The component contains ∞

[pf: • $\eta(s)$: continuous arc connecting a and b $\quad 0 \le s \le 1$
$\eta(s)$ does NOT intersect γ

$\Rightarrow \displaystyle\frac{1}{2\pi i}\int_\gamma \frac{1}{z-\eta(s)}\, dz$ is a continuous function, but its value is an integer $\Rightarrow n(\gamma, a) = n(\gamma, \eta(s)) = n(\gamma, b)$

• choose Δ = large disk containing γ

For $c \notin \Delta$, $n(\gamma, c) = 0 \Rightarrow n(\gamma, a) = 0$ $\qquad$ for a in the same component] rmk One can state some geometric condition on γ & a such that $n(\gamma, a) = 1$, for example [P. 116, lemma 2]

§5 Cauchy integral formula

recall • $\displaystyle\int_{\partial\Delta} g(z)\,dz = 0$

g: analytic on $\Delta \setminus \{\zeta_j\}$ with $\boxed{\lim_{z\to\zeta_j} (z-\zeta_j)\, g(z) = 0}$

• $\displaystyle\int_{\partial\Delta} \frac{1}{z-a}\,dz = 2\pi i$

What happens if the limit condition is not true?
For instance, consider $\dfrac{f(z)}{z-a}$ for analytic $f(z)$.

$\displaystyle\int_{\partial\Delta} \frac{f(z)}{z-a}\,dz = ?$ But $\dfrac{f(z)-f(a)}{z-a}$ satisfies the limit condition

$$\Rightarrow \int_{\cancel{\partial\Delta}\ \gamma} \frac{f(z)-f(a)}{z-a}\,dz \Rightarrow \int_{\cancel{\partial\Delta}\ \gamma} \frac{f(z)}{z-a}\,dz = \int_{\cancel{\partial\Delta}\ \gamma} \frac{f(a)}{z-a}\,dz$$

$$= 2\pi i \underbrace{\ }_{n(\gamma,a)} f(a)$$

Thm $f(z)$: analytic on some disk Δ.
$a \in \Delta$, γ: a closed curve in Δ not passing through a
Then, $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = n(\gamma, a)\, f(a)$

rmk The value of $f(z)$ at a is determined by its value on γ!

Moreover, we can move a, and the formula still holds

Assume $n(\gamma, w) = 1$ for any w in a small disk

$$\Rightarrow f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz$$

Suppose we can interchange derivative and integration

$$\Rightarrow f'(w) = \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial w}\left(\frac{f(z)}{z-w}\right)dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^2}\,dz$$

$$\Rightarrow f^{(n)}(w) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^{n+1}}\,dz$$

[We will prove the formula next week.
It turns out that the formula has a plethora of nontrivial consequences about analytic functions.]