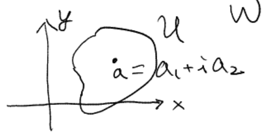
I. basics of analytic functions

§1 (complex) derivative

traditional approach : existence of complex derivative

<u>defn</u> [limit] $f : \mathcal{U} \subset \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{R}$ or $\mathbb{C}$, $a \in \mathcal{U}$



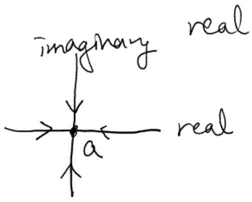
We say $\lim_{z \to a} f(z) = A$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $|f(z) - A| < \varepsilon$ $\forall z$ with $|z-a| < \delta$

Let us also replace the definition of derivative by complex numbers, and see what happens.

<u>defn</u> [complex derivative] $f : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ or $\mathbb{C}$ $a \in \mathcal{U}$

$f'(a)$ is defined to be $\lim_{z \to a} \dfrac{f(z) - f(a)}{z - a}$, provided the limit exists

e.g. If f is real-valued , we can take the limit along the real direction or the imaginary direction



$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \in \mathbb{R} \qquad (z - a = h \in \mathbb{R})$$

$$= \lim_{h \to 0} \frac{f(a+ih) - f(a)}{ih} \in i\mathbb{R} \qquad (z - a = ih \in i\mathbb{R})$$

Hence, if the limit exists , it must be zero.

In other words, it is not interesting to consider the complex derivative of a real-valued function.

§2 Cauchy - Riemann equation

<u>defn</u> [analytic / holomorphic] $f : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ is said to be analytic if $f'(a)$ exists $\forall a \in \mathcal{U}$.

Let us study the condition closely by using the real & imaginary part

Write $f(z) = u(z) + i\, v(z)$. Again, take the limits along $\mathbb{R}$ & $i\mathbb{R}$.

$$f'(a) = \lim_{h \to 0} \frac{u(a+h) + i\,v(a+h) - u(a) - i\,v(a)}{h} = \left.\frac{\partial u}{\partial x}\right|_a + i \left.\frac{\partial v}{\partial x}\right|_a = \left.\frac{\partial f}{\partial x}\right|_a$$

$h \in \mathbb{R}$ $\begin{cases} a+h \in \mathbb{C} \\ = (a_1 + h, a_2) \in \mathbb{R}^2 \\ a + ih \in \mathbb{C} \\ = (a_1, a_2 + h) \in \mathbb{R}^2 \end{cases}$

$$= \lim_{h \to 0} \frac{u(a+ih) + i\,v(a+ih) - u(a) - i\,v(a)}{ih} = -i \left.\frac{\partial u}{\partial y}\right|_a + \left.\frac{\partial v}{\partial y}\right|_a = -i \left.\frac{\partial f}{\partial y}\right|_a$$

Therefore, if $f'(a)$ exists $\forall a \in \mathcal{U}$

then its real and imaginary parts must obey

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

the Cauchy - Riemann equation (CR eqn)

§3 consequences of CR eqn

1° $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial(u,v)}{\partial(x,y)}$ (the Jacobian)

2° Try to cancel v from CR eqn

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}$ i.e. $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, Similarly $\Delta v = 0$

Thus, u & v are harmonic function, i.e. they solves the Laplace equation $\Delta(-) = 0$.

rmk if u & v solves CR eqn (thus $\Delta u = 0 = \Delta v$), v is said to be the conjugate harmonic function of u.

3° $f = u + iv$ is analytic with $f' \in \mathcal{C}^0$

$\Leftrightarrow$ u & $v \in \mathcal{C}^1$ solve the CR eqn

pf: $\Rightarrow$ already explained

$\Leftarrow$ $u(a_1+h_1, a_2+h_2) - u(a_1,a_2) = \frac{\partial u}{\partial x}\Big|_a \cdot h_1 + \frac{\partial u}{\partial y}\Big|_a \cdot h_2 + o\left(\sqrt{h_1^2+h_2^2}\right)$

$v(a_1+h_1, a_2+h_2) - v(a_1,a_2) = -\frac{\partial u}{\partial y}\Big|_a \cdot h_1 + \frac{\partial u}{\partial x}\Big|_a \cdot h_2 + o\left(\sqrt{h_1^2+h_2^2}\right)$

It is not hard to show that $f'(a) = \frac{\partial u}{\partial x}\Big|_a - i\frac{\partial u}{\partial y}\Big|_a$ ※

4° [viewpoint from change of variable]

Introduce $\bar z = x - iy$. $\Rightarrow x = \frac{z+\bar z}{2}$ $y = \frac{z - \bar z}{2i}$

We can "forget" that z & $\bar z$ are conjugate to each other.

Then. $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$ (pretend the chain rule works as well here)

$\frac{\partial f}{\partial \bar z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$

Moreover,
- $\frac{\partial f}{\partial \bar z} = 0 \Leftrightarrow$ u & v obeys the CR eqn.
- if $\frac{\partial f}{\partial \bar z} = 0$ (f is analytic), $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} = f'(z)$

rmk If we do $f(x,y) = f\left(\frac{z+\bar z}{2}, \frac{z-\bar z}{2i}\right)$, analytic function is independent of $\bar z$ (by forgetting $\bar z$ is the complex conjugate of z)

§4 rational functions

1° [polynomial] $P(z) = a_0 + a_1 z + \cdots + a_n z^n$: any degree n polynomial in z is analytic $(a_n \neq 0)$

check: By fundamental theorem of algebra, $P(z) = a_n(z-\alpha_1)\cdots(z-\alpha_n)$

$z - \alpha_1 = (x - \mathrm{Re}\,\alpha_1) + i(y - \mathrm{Im}\,\alpha_1)$, i.e. $u = x$, $v = y$ is clearly analytic.

It is not hard to show that the product of analytic functions is still analytic. Furthermore, $(fg)' = f'g + fg'$

(In fact, the product rule also implies that z^k is analytic $\forall k \in \mathbb{N}$)

2° [order of zeros] $P(z)$: polynomial of degree n

α = zero of $P(z)$, i.e. $P(\alpha)=0$ is said to have order $h \in \mathbb{N}$ if $P(z) = (z-\alpha)^h P_h(z)$ with $P_h(\alpha) \neq 0$

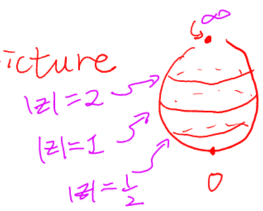
3° [rational function & pole] a rational function $R(z)$ is the quotient of two polynomials. $\dfrac{P(z)}{Q(z)}$. (we usually assume that P & Q have no common zeros)

It is not hard to show that $R(z)$ is analytic outside the zeros of Q, and $R'(z) = \dfrac{P'(z)Q(z) - P(z)Q'(z)}{Q(z)^2}$

The zeros of $Q(z)$ are called the poles of $R(z)$.
In a way, $R(z) = \infty$ at the poles.

4° [including ∞] It is more convenient to consider rational functions on the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (some people use $\bar{\mathbb{C}}$ or $\mathbb{C}_\infty$)

picture



We use the following definition for rational functions.

- value : we say that $R(\alpha) = \infty$ for some $\alpha \in \mathbb{C}$ if $\lim_{z \to \alpha} 1/R(z) = 0$
- domain : the value of $R(z)$ at ∞ is defined to be the value of $R(\frac{1}{w})$ at $w=0$

i) α : pole of $R(z)$, $R(\alpha) := \lim_{z\to\alpha} \dfrac{\overset{\to \text{finite}}{P(z)}}{\underset{\to 0}{Q(z)}} = \infty$
We define the order of a pole to be the order of corresponding zero of $Q(z)$.

ii) the value of R at ∞

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}$$

substitute z by $1/w$ and let $w \to 0$

$$\rightsquigarrow R\left(\frac{1}{w}\right) = \frac{a_0 + a_1 w^{-1} + \cdots + a_n w^{-n}}{b_0 + b_1 w^{-1} + \cdots + b_m w^{-m}} = w^{m-n} \frac{\overbrace{a_0 w^n + \cdots + a_n}^{\xrightarrow{w\to 0}\, a_n}}{\underbrace{b_0 w^m + \cdots + b_m}_{\xrightarrow{w\to 0}\, b_m}}$$

$$\lim_{w\to 0} R\left(\frac{1}{w}\right) = \begin{cases} 0 & \text{if } m>n : R(z) \text{ has zero of order } m-n \\ a_n/b_m & \text{if } m=n : R(\infty) := a_n/b_m \\ \infty & \text{if } m<n : R(z) \text{ has pole of order } n-m \end{cases}$$

5° What is it good for? Let us count zeros and poles on $\hat{\mathbb{C}}$

$\#\{z \in \hat{\mathbb{C}} \mid R(z) = 0\} = \max\{n, m\}$

$$\left.\begin{array}{l} z \in \mathbb{C} \Rightarrow P(z) = 0 \Rightarrow n \text{ of them} \\ z = \infty \Rightarrow \text{only happens when } m > n \Rightarrow m-n \end{array}\right\} \Rightarrow \max\{n, m\}$$

$\#\{z \in \hat{\mathbb{C}} \mid R(z) = \infty\} = \max\{n, m\}$

$$\left.\begin{array}{l} z \in \mathbb{C} \Rightarrow Q(z) = 0 \Rightarrow m \text{ of them} \\ z \in \hat{\mathbb{C}} \Rightarrow \text{only happens when } m < n \Rightarrow n-m \end{array}\right\} \Rightarrow \max\{n, m\}$$

Similarly $\#\{z \in \hat{\mathbb{C}} \mid R(z) = \alpha\} = \max\{n, m\} \quad \forall \alpha$

$\#$: counting multiplicities (we abuse the notation here)

This number is called the order of the rational function $R(z)$, and every equation $R(z) = \alpha$ has exactly this number of solutions.

6° [application: deriving partial fractions]

recall write $\frac{f(x)}{g(x)}$ as polynomials $+ \frac{a_1}{x-\alpha_1} + \cdots + \frac{a_m}{x-\alpha_m}$

$$R(z) = \frac{P(z)}{Q(z)} = G(z) + \underbrace{\frac{\tilde{P}(z)}{Q(z)}}_{} \quad H(z) =: \frac{\tilde{P}(z)}{Q(z)}$$

where $G(z)$: polynomial without constant term

$\deg \tilde{P} \le \deg Q$

not necessary → assume $\deg P > \deg Q$

i) Namely. $H(\infty) \neq \infty$
$G(\infty) = \infty$ $\quad$ G captures how $R \to \infty$ as $z \to \infty$

ii) Strategy. get the significant terms at all poles
$\Rightarrow$ a rational function doesn't map to ∞
$\Rightarrow$ constant

iii) $\{\beta_1, \cdots, \beta_q\}$ (distinct) finite poles of $R(z)$

We can move β_j to ∞, the apply the same trick to get the significant term.

$R(\beta_j + \frac{1}{w})$: rational function in w with a pole at ∞

By the same token, write it as $G_j(w) + H_j(w)$

$H_j(\infty) \in \mathbb{C}$

$$\Rightarrow R(z) = G_j\left(\frac{1}{z-\beta_j}\right) + H_j\left(\frac{1}{z-\beta_j}\right)$$

← polynomial in $\frac{1}{z-\beta_j}$: $C_1 \frac{1}{z-\beta_j} + \cdots + C_\ell \left(\frac{1}{z-\beta_j}\right)^\ell$

iv) Consider the rational function

$$\tilde{R}(z) = R(z) - G(z) - \sum_{j=1}^{q} G_j\left(\frac{1}{z-\beta_j}\right)$$

- It has no poles other than $\beta_1, \cdots, \beta_q$ and ∞.
- At β_1. $\tilde{R}(\beta_1) = \underbrace{R(\beta_1) - G_1(\infty)}_{= H_1(\infty) \neq \infty} - G(\beta_1) - \sum_{j=2}^{q} G_j\left(\frac{1}{\beta_1-\beta_2}\right) \neq \infty$
- At ∞, $\tilde{R}(\infty) = \underbrace{R(\infty) - G(\infty)}_{= H(\infty) \neq \infty} - \sum_{j=1}^{q} G_j(0) \neq \infty$

v) Therefore, $R(z) = G(z) + \sum_{j=1}^{q} G_j\left(\frac{1}{z-\beta_j}\right) + \text{Const.}$

e.g. $R(z) = \frac{1}{z(z+2)^2}$ $\quad \beta_1 = 0 \quad \beta_2 = -2$

@ β_1 $\quad z = \frac{1}{w} \quad R\left(\frac{1}{w}\right) = \frac{w^3}{(2w+1)^2} = \underbrace{\frac{w}{4}}_{= G_1(w)} + \frac{(\cdots)}{(2w+1)^2}$

@ β_2 $\quad z = -2 + \frac{1}{w} \quad R\left(-2 + \frac{1}{w}\right)$

$$= \frac{w^3}{(1-2w)} = \underbrace{\left(-\frac{1}{2}w^2 - \frac{1}{4}w\right)}_{= G_2(w)} + \frac{(\cdots)}{1-2w}$$

$$\Rightarrow \frac{1}{z(z+2)^2} = \frac{1}{4z} - \frac{1}{2}\frac{1}{(z+2)^2} - \frac{1}{4}\frac{1}{z+2} + \underbrace{C_0}_{= 0 \text{ here}}$$

※

§5 power series (general examples of analytic functions)

Power series, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, is a natural generalization of polynomial. It not surprising that a power series is analytic on where it converges "nicely."

1° <u>Thm</u> [Abel] For $\sum_{n=0}^{\infty} a_n z^n$, $\exists$ $R \geq 0$ (could be ∞) : the radius of convergence with the following significance:

i) It converges absolutely (i.e. $\sum_{n=0}^{\infty} |a_n z^n|$ converges) $\forall z$ with $|z|<R$
Given any ρ with $0 \leq \rho < R$, the convergence is uniformly for $|z| \leq \rho$.

ii) It diverges for $|z| > R$

iii) When $|z|<R$, it is an analytic function, whose derivative can be obtained by termwise differentiation.

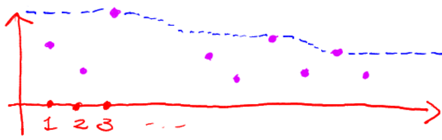
pf: Idea: compare to geometric series

$|a_n| R^n \leq 1$ for n: sufficiently large

$\Leftrightarrow \sqrt[n]{|a_n|} \leq \frac{1}{R}$

$\{b_n\}$: real-valued sequence

$\limsup = \lim_{n\to\infty} \sup_{l \geq n} \{b_l\}$



Let $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

For i): If $|z| < R$, choose $\rho \in (|z|, R)$ $\qquad \frac{1}{R} < \frac{1}{\rho}$

$\Rightarrow \exists N$ such that $\sqrt[n]{|a_n|} < \frac{1}{\rho}$ $\quad \forall n \geq N$

$|a_n z^n| < \left(\frac{|z|}{\rho}\right)^n \Rightarrow$ DONE

The proof for uniform convergence is the same

For ii): If $|z| > R$, choose $\rho \in (R, |z|)$ $\qquad \frac{1}{R} > \frac{1}{\rho}$

$\limsup_{n\to\infty} \sqrt[n]{|a_n|} > \frac{1}{\rho}$

$\Rightarrow \exists$ subsequence n_j $\rightarrow$ $|a_{n_j} z^{n_j}| > \left(\frac{|z|}{\rho}\right)^{n_j}$ greater than 1

Note that R is the only number satisfying i) and ii)

For iii): Consider $\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$

<u>claim</u> Its radius of convergence R' is equal to R

Since $\lim_{n\to\infty} \sqrt[n]{n+1} = 1$, $\limsup_{n\to\infty} \sqrt[n]{(n+1)a_{n+1}} = \limsup_{n\to\infty} \sqrt[n]{a_{n+1}}$, and we may consider $\sum_{n=0}^{\infty} a_{n+1} z^n$

$a_0 + a_1 z + a_2 z^2 + \cdots = a_0 + z(a_1 + a_2 z^2 + \cdots) \Rightarrow R' \leq R$

$a_1 + a_2 z^2 + \cdots = (a_0 + a_1 z + \cdots) - a_0 \big/ z \Rightarrow R \leq R'$

We now examine the derivative:

$$\frac{\sum a_n z^n - \sum a_n z_0^n}{z - z_0} - \sum n a_n z_0^{n-1} = (\text{sum from } n=1 \text{ to } m)$$

$$+ \sum_{n=m+1}^{\infty} n a_n z_0^{n-1} + \sum_{n=m+1}^{\infty} a_n \left(\underbrace{z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1}}_{|\cdots| \le n\rho^n}\right)$$

$$z^n - z_0^n = (z - z_0)(z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1})$$

Assume that $z \neq z_0$, $|z|, |z_0| < \rho < R$

$\Rightarrow \forall \varepsilon > 0$ $\exists m$ such that the remainder terms $< \varepsilon$

The leading term tends to zero as $z \to z_0$ #

2° basic example: the exponential function

$$e^z = 1 + z + \frac{1}{2} z^2 + \cdots + \frac{1}{n!} z^n + \cdots$$

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0 \Rightarrow$ the radius of convergence is ∞

It is not hard to prove the usual properties of exp.

Note that $(e^z)' = e^z$.

e.g. $e^{a+b} = e^a \cdot e^b$

pf: $(e^z \cdot e^{c-z})' = e^z \cdot e^{c-z} + e^z \cdot (-e^{c-z}) = 0$

$\Rightarrow e^z \cdot e^{c-z} = e^0 \cdot e^c = e^c$

let $z = a$, $c = b + a \Rightarrow e^a \cdot e^b = e^{a+b}$ #

In particular $e^{x+iy} = e^x \cdot e^{iy} \underset{\text{by power series}}{=} e^x \cdot (\cos x + i \sin y)$

We can also define sine & cosine for variable $z \in \mathbb{C}$