COMPLEX ANALYSIS
MIDTERM

THURSDAY, NOVEMBER 12

(1) [8 points] True/False questions, no justifications needed.

(a) Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. There exists a rational function $q(z)$ such that $\#\{q^{-1}(0)\} = 2$ and $\#\{q^{-1}(\infty)\} = 0$ (regard $q(z)$ as a map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, and count zeros with multiplicity).

$\boxed{\text{F}}$ As a consequence of the fundamental theorem of algebra, $\#\{q^{-1}(u)\}$ is the same for any $u \in \hat{\mathbb{C}}$.

(b) There exists an analytic function $f(z)$ defined on $\{z \in \mathbb{C} \mid |z| < 2\}$ such that $f^{(k)}(0) = 1$ for any $k \geq 0$.

$\boxed{\text{T}}$ $f(z) = \exp(z)$ does the job.

Originally, I planned have the following function $f(z) = 1/(1-z)$, which has coefficient 1 but $f^{(k)}(0) = k!$.

(By Abel's theorem, the power series has radius of convergence 1, and $\sum_{k\geq 0} z^k$ defined an analytic function for $|z| < 1$.) The function $1/(1-z)$ has that property. Since an analytic function is determined by its value and all the derivatives at one point, the function must be $1/(1-z)$, which cannot be analytic on $\{z \in \mathbb{C} \mid |z| < 2\}$

(c) Consider the function $f(z) = \cosh \dfrac{1}{z}$. There exist positive constants ϵ and M so that $|f(z)| > M$ for any $0 < |z| < \epsilon$.

$\boxed{\text{F}}$ The point $z = 0$ is an essential singularity. The image of any small neighborhood is always dense in $\mathbb{C}$.

(d) Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Suppose that $f : D \to D$ is analytic with $f(0) = 0$. Then, $|f(z)| \leq |z|$ for any $z \in D$.

$\boxed{\text{T}}$ This is Schwarz lemma.

(e) There does not exist an entire function whose value on the *positive real axis* is $\int_0^\infty e^{-t} t^{z-1} \mathrm{d}t$.

$\boxed{\text{T}}$ If the function exists, it coincides with the gamma function $\Gamma(z)$ on the positive real axis, which is not discrete. Therefore, the function must be $\Gamma(z)$ (or coincides with $\Gamma(z)$ except on the poles of $\Gamma(z)$), but such a function cannot be analytic everywhere.

(f) There does not exist an entire function whose value on $n \in \mathbb{N}$ is $(n-1)!$.

$\boxed{\text{F}}$ The condition is on a discrete set. One can construct such a function by modifying the Weiestrass product construction, see [Ahlfors, #1 of p.197]

(g) There exists a meromorphic function whose pole is exactly $n \in \mathbb{N}$ with the singular part $\sum_{k=1}^{n} \frac{\exp(2^k)}{(z-n)^k}$.

$\boxed{\text{T}}$ It follows directly from the theorem of Mittag-Leffler. The singular part does not matter.

(h) Let $\mathcal{B}$ be the set of entire functions functions with $f(0) = 1$ and $|f(z)| \leq 999$ when $|z| = 100$. For any $n \in \mathbb{N}$, there always exists $f(z) \in \mathcal{B}$ which has exactly n zeros in $\{z \in \mathbb{C} \mid |z| < 50\}$.

$\boxed{\text{F}}$ Jensen's formula gives a bound for the number of zeros.

(2) [5 points] Suppose that $f(z)$ is an entire function obeying $|f(z)| \leq n^{\frac{3}{4}}$ when $|z| = n \in \mathbb{N}$. Prove that $f(z)$ is a constant function. (Hint: There are many ways to do it. You can show that $f^{(k)}(0) = 0$ for any $k \in \mathbb{N}$, or $f(z_1) = f(z_2)$ for any $z_1, z_2 \in \mathbb{C}$, etc.)

[method 1] It follows from the Cauchy integral formula that

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=n} \frac{f(z)}{z^{k+1}} \mathrm{d}z \quad \text{for any } n \in \mathbb{N} \ ,$$

and then

$$|f^{(k)}(0)| \leq k! \frac{n^{\frac{3}{4}}}{n^{k+1}} n \ .$$

It is clear that the right hand side tends to zero as $n \to \infty$ for any fixed $k \geq 1$. Thus, $f^{(k)}(0) = 0$ for any $k \geq 1$, and $f(z)$ must be a constant function.

[method 2] Given any $z_1, z_2 \in \mathbb{C}$, consider $n \in \mathbb{N}$ so that n is greater than $2|z_1|$, $2|z_2|$. By Cauchy integral formula

$$f(z_1) - f(z_2) = \frac{1}{2\pi i} \int_{|z|=n} \left(\frac{1}{z - z_1} - \frac{1}{z - z_2} \right) f(z) \mathrm{d}z$$

$$\Rightarrow \ |f(z_1) - f(z_2)| \leq \frac{1}{2\pi} \left(\frac{2}{n} \frac{2}{n} |z_1 - z_2| \right) n^{\frac{3}{4}} (2\pi n) \ .$$

It is clear that the right hand side tends to zero as $n \to \infty$. Hence, $f(z_1) = f(z_2)$ for any $z_1, z_2 \in \mathbb{C}$.

(3) [5 points] How many roots does the equation $z^{2015} + 2z^{304} + 11z^{12} + z^4 + z^3 + z^2 + z + 1 = 0$ have in $\{z \in \mathbb{C} \mid |z| < 1\}$? Justify your answer.

We apply Rouché's theorem. Let $f(z) = z^{2015} + 2z^{304} + 11z^{12} + z^4 + z^3 + z^2 + z + 1$ and $g(z) = 11z^{12}$. By the triangle inequality,

$$|f(z) - g(z)| \leq 8 < |g(z)| \quad \text{when } |z| = 1 \ .$$

Hence, $f(z)$ and $g(z)$ have the same number of roots in the unit disk, and there are 12 roots.

(4) [6 points] Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} \mathrm{d}x \quad \text{for } 0 < a < 1 \ .$$

(Hint: e^z has a period of $2\pi i$.)

Let $R_M = \{z = x + iy \in \mathbb{C} \mid -M \leq x \leq M, 0 \leq y \leq 2\pi\}$. The function $f(z) = \frac{e^{az}}{1+e^z}$ has only one pole in the region R_M. It is at $z = \pi i$ with residue $-e^{\pi a i}$. It follows that $\int_{\partial R_M} f(z)\mathrm{d}z = -2\pi i e^{\pi a i}$. It is not hard to show that

$$\lim_{M\to\infty} \int_{\partial R_M} f(z)\mathrm{d}z = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\mathrm{d}x - \int_{-\infty}^{\infty} \frac{e^{2\pi a i}e^{ax}}{1+e^x}\mathrm{d}x = (1 - e^{2\pi a i}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\mathrm{d}x \ .$$

Then,

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\mathrm{d}x = \frac{-2\pi i e^{\pi a i}}{1 - e^{2\pi a i}} = \frac{2\pi i}{e^{\pi a i} - e^{-\pi a i}} = \frac{\pi}{\sin(\pi a)} \ .$$

(5) [$6 = 2 + 4$ points] Consider the function

$$f(z) = \frac{e^{2\pi z} - 1}{z} \ .$$

(a) Show that $z = 0$ is a removable singularity, and determine $f(0)$.

Since $\lim_{z\to 0} |z|\,|f(z)| = 0$, $z = 0$ is a removable singularity. Its value is

$$\lim_{z\to 0} \frac{e^{2\pi z} - 1}{z} = 2\pi \frac{\partial}{\partial w}\Big|_{w=0} e^w = 2\pi \ .$$

(b) Construct the Weierstrass product development of $f(z)$, and find its genus. You shall briefly explain the convergence of your expression. You can simply invoke the theorem of Weierstrass, or derive the estimate by hand.

(Hint: You may need the following formula:

$$\frac{2\pi e^{2\pi z}}{e^{2\pi z} - 1} = \pi + \frac{1}{z} + \sum_{n\neq 0} \left(\frac{1}{z - in} + \frac{1}{in}\right) .)$$

The zeros of $f(z)$ are nonzero integers, each one is of order 1. Since $\sum_{n\neq 0}\left(\frac{1}{|n|}\right)^{h+1}$ diverges for $h = 0$ and converges for $h = 1$, the canonical product is

$$\prod_{n\neq 0}(1-\frac{z}{in})\exp(\frac{z}{in})\ .$$

The function $f(z)$ shall be

$$e^{g(z)}\prod_{n\neq 0}(1-\frac{z}{in})\exp(\frac{z}{in})$$

for some entire function $g(z)$. The log-derivative reads

$$\frac{2\pi e^{2\pi z}}{e^{2\pi z}-1}-\frac{1}{z}=g'(z)+\sum_{n\neq 0}\frac{z}{in(z-in)}\ .$$

By combining it with the hint, we find that $g'(z) = \pi$ and $g(z) = \pi z + c_0$. The constant c_0 can be found by evaluating at $z = 0$, and $c_0 = \log 2\pi$. It follows that $f(z)$ has genus 1.

(6) [4 points] Let $f(z)$ be a non-constant entire function of *finite* order. Prove that the image of f can miss at most one value in $\mathbb{C}$. (Hint: You may assume f misses α and β. What can you say about the entire function $f(z)-\alpha$? Could it be possible that $f(z)-\alpha$ never equals to $\beta-\alpha$?)

Consider the function $f(z)-\alpha$. It is of the same order as $f(z)$. (This can be seen by using the equivalent condition that for any $\epsilon > 0$, $|f(z)| \le \exp(|z|^{\lambda+\epsilon})$ for $|z|$ being sufficiently large.) We know $f(z)-\alpha$ is a nowhere zero entire function, and thus can be written as $e^{g(z)}$ for another entire function $g(z)$. By Hadamard's theorem, g must be a polynomial, and its degree is the maximal integer no greater than the order of f.

But the fundamental theorem of algebra implies that g assumes every value, in particular $\log(\beta-\alpha)$ (the branch does not matter here). Thus, f cannot miss the value β.

In fact, it is true for *any* non-constant entire function. It is known as the Little Picard Theorem: the image of a non-constant entire function is either $\mathbb{C}$ or $\mathbb{C}$ minus a point.