COMPLEX ANALYSIS
HOMEWORK 14

DUE: TUESDAY, DECEMBER JANUARY 5

The normal family in this homework set is in the sense of meromorphic functions.

(1) Let U be an open and connected subset of $\mathbb{C}$, and $g(z)$ is a non-constant analytic function on U. Denote $g(U)$ by V. Let $\mathcal{H}$ be a family of meromorphic function on V. Prove that $\mathcal{H}$ is normal on V if and only if $\{f \circ g \mid f \in \mathcal{H}\}$ is normal on U.

(2) Let $\mathcal{S} = \{f(z) : \text{injective analytic function on } D \mid f(0) = 0 \text{ and } f'(0) = 1\}$.

  (a) Show that there exists $\kappa > 0$ such that the image of any $f \in \mathcal{S}$ contains the open disk centered at 0 with radius κ. You may prove by contradiction. Suppose there is a sequence $\{f_n\} \in \mathcal{S}$ and $\{w_n\} \in \mathbb{C}$ such that $w_n \notin f_n(D)$ and $w_n \to 0$. Try to construct a normal family of functions from $\{f_n\}$ and $\{w_n\}$.

  (b) Examine the function $f(z) = z/(1-z)^2$, and conclude that $\kappa \leq 1/4$.

(3) Let Ω be an open and connected subset of $\mathbb{C}$, and $\mathcal{G}$ be the family of all *injective* analytic functions on Ω.

  (a) Show that $\mathcal{G}$ is not normal.

  (b) Show that the family of functions in $\mathcal{G}$ that omit 0 is normal.

  (c) Show that the family of derivatives of functions in $\mathcal{G}$ is normal. That is to say, $\{f\text{: analytic on } \Omega \mid f = h' \text{ for some } h \in \mathcal{G}\}$ is normal.

  (d) Bonus Show that for fixed $z_0 \in \Omega$ and $M > 0$, $\{f \in \mathcal{G} \mid |f'(z_0)| \leq M\}$ is normal.