COMPLEX ANALYSIS
HOMEWORK 13

DUE: TUESDAY, DECEMBER 29

(1) Let Ω be an open, connected and bounded region in $\mathbb{C}$ whose boundary consists of smooth curves. The Green's theorem says that

$$\iint_{\Omega} (q_x - p_y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} p\,\mathrm{d}x + q\,\mathrm{d}y\ .$$

for $p, q \in \mathcal{C}^2(\bar{\Omega})$.

(a) Prove Green's first identity:

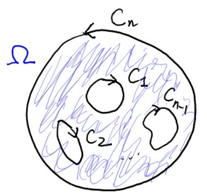
$$\iint_{\Omega} (v\Delta u + \nabla u \cdot \nabla v)\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} v\frac{\partial u}{\partial n}\mathrm{d}s$$

where n is the unit outer normal, and s is the arc-length parameter for $\partial\Omega$.

(b) Prove Green's second identity:

$$\iint_{\Omega} (v\Delta u - u\Delta v)\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right)\mathrm{d}s\ .$$

(2) Consider a multiply-connected region as shown in the picture. Assume there are more than three boundary components, and each boundary curve C_j is the image of the unit circle under some conformal map (defined on a open neighborhood of the unit circle). Let $w_j(z)$ be the harmonic function with the boundary value $w_j(z) = 1$ when $z \in C_j$ and $w_j(z) = 0$ for $z \in C_{k\neq j}$.



By the reflection principle, the functions $w_j(z)$ can be regarded as harmonic functions defined on $\bar{\Omega}$. As explained in class, let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the unique solution to the equations

$$\sum_{j=1}^{n-1} \lambda_j \int_{C_k} \frac{\partial w_j}{\partial n}\mathrm{d}s = \begin{cases} 2\pi & \text{for } k = 1\ , \\ -2\pi & \text{for } k = n\ , \\ 0 & \text{for } 1 < k < n\ . \end{cases}$$

Then, set $u(z)$ to be $\sum_{j=1}^{n-1} \lambda_j\, w_j(w)$, and $v(z)$ to be its conjugate harmonic. Namely, $v(z)$ is a multiple-valued function given by the line integral of $-u_y\mathrm{d}x + u_x\mathrm{d}y$. By construction, $F = \exp(u + iv)$ is a well-defined analytic function on $\bar{\Omega}$.

(a) Show that
$$\int_{C_j} \frac{\partial w_k}{\partial n} \mathrm{d}s = \int_{C_k} \frac{\partial w_j}{\partial n} \mathrm{d}s$$
for any $j, k \in \{1, \dots, n-1\}$.

(b) Show that
$$\frac{1}{2\pi i} \int_{C_1} \frac{F'(z)}{F(z) - w_0} \mathrm{d}z = \begin{cases} 1 & \text{when } |w_0| < e^{\lambda_1} \ , \\ \frac{1}{2} & \text{when } |w_0| = e^{\lambda_1} \ , \\ 0 & \text{when } |w_0| > e^{\lambda_1} \ . \end{cases}$$

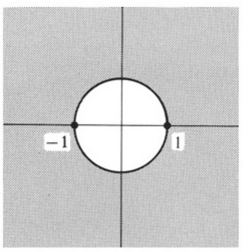
(c) Show that
$$\frac{1}{2\pi i} \int_{C_n} \frac{F'(z)}{F(z) - w_0} \mathrm{d}z = \begin{cases} -1 & \text{when } |w_0| < 1 \ , \\ -\frac{1}{2} & \text{when } |w_0| = 1 \ , \\ 0 & \text{when } |w_0| > 1 \ . \end{cases}$$

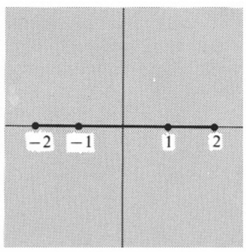
(d) Show that
$$\frac{1}{2\pi i} \int_{C_k} \frac{F'(z)}{F(z) - w_0} \mathrm{d}z = 0$$
for $k \in \{2, \dots, n-1\}$ and any w_0.

With these items, it follows that $1 < \lambda_1$ and
$$\frac{1}{2\pi i} \sum_{k=1}^{n} \int_{C_k} \frac{F'(z)}{F(z) - w_0} \mathrm{d}z = \begin{cases} \frac{1}{2} & \text{when } |z| = 1 \text{ or } e^{\lambda_1} \ , \\ 1 & \text{when } 1 < |z| < e^{\lambda_1} \ , \\ 0 & \text{otherwise.} \end{cases}$$

Then one can figure out the image of $F(z)$ by the argument principle and the construction of F.

(3) (a) Show that the function $z \mapsto z + \frac{1}{z}$ defines a conformal equivalence between the region outside the unit circle onto the plane from which the segment $[-2, 2]$ has been deleted.





(b) What is the image of the unit circle under this mapping? Use polar coordinates.

(c) In polar coordinates, if $w = z + \frac{1}{z} = u + iv$, then
$$u = \left(r + \frac{1}{r}\right) \cos\theta \quad \text{and} \quad v = \left(r - \frac{1}{r}\right) \sin\theta \ .$$
Show that the circle $r = c$ with $c > 1$ maps to an ellipse with major axis $c + \frac{1}{c}$ and minor axis $c - \frac{1}{c}$. Show that the radial lines $\theta = c$ map onto quarters of hyperbolas.