COMPLEX ANALYSIS
HOMEWORK 12

DUE: TUESDAY, DECEMBER 22

(1) Let $f(z)$ be an entire function with
$$\lim_{z\to\infty} \frac{\operatorname{Re} f(z)}{z} = 0 \ .$$
Show that $f(z)$ is a constant function. You may need (66) on p.168 of Ahlfors.

(2) Let $f(z)$ be an analytic function on $\{z \in \mathbb{C} \mid |z| > M\}$ for some $M >> 1$.
   (a) Show that $f(z) = f_1(z) + f_2(z)$ where $\lim_{z\to\infty} f_1(z) = 0$ and $f_2(z)$ is an entire function. You will need the Cauchy integral formula, and the proof of the Laurent series expansion might help you.
   (b) Suppose that
   $$\lim_{z\to\infty} \frac{\operatorname{Re} f(z)}{z} = 0$$
   Show that $\lim_{z\to\infty} f(z)$ exists.

(3) Suppose that $u(z)$ is a harmonic function on $\Omega = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ with $\lim_{z\to 0} z\, u(z) = 0$.
   (a) Prove that $\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\mathrm{d}\theta$ is a linear function of $\log r$. Namely, there exists constants α and β such that $\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\mathrm{d}\theta = \alpha \log r + \beta$.
   (b) Prove that $u(z) = \alpha \log |z| + u_0(z)$ for some harmonic function $u_0(z)$ on D. At least, $u_0(z)$ is analytic on Ω, you can start by showing that it is the real part of an analytic function on Ω, and then prove that 0 is a removable singularity.

(4) Consider $\Omega = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Let $f(\zeta) = 0$ for $|\zeta| = 1$ and $f(0) = 1$. Explain that the Dirichlet problem on Ω with boundary condition $f(\zeta)$ has no solution.

(5) Let Ω be an open and connected subset of $\mathbb{C}$, and $u(z)$ is a $\mathcal{C}^2$ function on Ω. Suppose that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geq 0$. Show that u is subharmonic.

(6) [Bonus] Let $u(z)$ be a harmonic function on $\{z \in \mathbb{C} \mid a < |z| < 1\}$ for some $a \in (0,1)$ with $\lim_{z\to\zeta} u(z) = 0$ for any $|\zeta| = 1$. Prove that $u(z)$ can be extended across $|\zeta| = 1$ as a harmonic function.