COMPLEX ANALYSIS
HOMEWORK 11

DUE: TUESDAY, DECEMBER 15

(1) Let k be a real number with $0 < k < 1$. Consider the following conformal map from $\mathbb{H}$ to a rectangle:

$$g(w) = \int_0^w \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}} .$$

Let

$$K = \int_{-1}^{1} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} ,$$

$$K' = \int_{1}^{\frac{1}{k}} \frac{\mathrm{d}t}{\sqrt{(t^2-1)(1-k^2t^2)}} .$$

As explained in [A, p.240], the image of g is the rectangle R_0 with vertex $K/2, -K/2, -K/2+iK'$ and $K/2+iK'$. By applying the Schwarz reflection principle, the inverse map $f : R \to \mathbb{H}$ can be extended to a doubly periodic meromorphic function on $\mathbb{C}$, with period $2K$ and $2K'$.

(a) Prove that $g(\infty) = iK'$.

(b) Show that $K = K'$ if and only if $k = (\sqrt{2}-1)^2$.

(c) Show that $f(z)$, $f(z+K)$ and $f(z+K')$ are odd functions of z; $f(z+K/2)$ and $f(z+K/2+K'/2)$ are even functions.

(2) Show that the conformal mapping of the unit disk onto a parallel strip, $\{z \in \mathbb{C} \mid 0 < \operatorname{Im} z < \pi i\}$, or onto a half strip with two right angles, $\{z \in \mathbb{C} \mid 0 < \operatorname{Im} z < \pi i \, , \operatorname{Re} z > 0\}$, can be obtained as special cases of the Schwarz–Christoffel formula.

(3) For any $r > 1$, denote the annulus $\{z \in \mathbb{C} \mid 1 < |z| < r\}$ by A_r. Prove that A_{r_1} is conformally equivalent to A_{r_2} if and only if $r_1 = r_2$. You may try to apply the Schwarz reflection principle consecutively to obtain a map from $\mathbb{C}\backslash\{0\}$ to itself.

(4) Given any three distinct points on ∂D, w_1, w_2 and w_3, show that there exists a map $f \in \operatorname{Aut}(D)$ such that $f(w_1) = 1$, $f(w_2) = i$ and $f(w_3) = -1$. (Assume w_1, w_2 and w_3 are in a counterclockwise order.)