COMPLEX ANALYSIS
HOMEWORK 10

DUE: TUESDAY, DECEMBER 8

(1) Let Ω and Ω' be open and connected sets in $\mathbb{C}$. Let $f : \Omega \to \Omega'$ be bijective and continuous. Moreover, $f^{-1} : \Omega' \to \Omega$ is also continuous. (Namely, f is a homeomorphism.) Let $\{z_n\}$ be a sequence in Ω which converges to a point $z \in \partial\Omega$. Suppose that $w = \lim_{n\to\infty} f(z_n)$ exists. Prove that $w \in \partial\Omega'$. (It suffices to show that $w \notin \Omega'$.)

(2) (a) Let $\Omega \subset \mathbb{C}$ be an open and connected set, and p is a point in Ω. Suppose that f is an analytic function on $\Omega\backslash\{p\}$ whose image $\Omega' = f(\Omega\backslash\{p\})$ is bounded. Show that f has a removable singularity at p. Moreover, if f is injective, show that $f(p) \in \partial\Omega'$.

(b) Let $\Omega_{r,R} = \{z \in \mathbb{C} \mid r < |z| < R\}$. Prove that $\Omega_{0,1}$ is not conformally equivalent to $\Omega_{r,R}$ for any $0 < r < R$.

(3) Let $\Omega \subset \mathbb{C}$ be a simply connected open set such that $\Omega \neq \mathbb{C}$ and $\bar{z} \in \Omega$ for any $z \in \Omega$. Let a be a point in $\Omega \cap \mathbb{R}$. Suppose that f is the unique bijective analytic map from Ω to D with $f(a) = 0$ and $f'(a) > 0$. Let $\Omega_+ = \{z \in \Omega \mid \operatorname{Im} z > 0\}$. Show that $f(\Omega_+)$ must lie entirely above or below the real axis.

(4) Let s be a real number within $[0, 1)$. Let U_s be the open set obtained from deleting the segment $[s, 1)$ from D. Construct a conformal equivalence from U_s to D. You may use the following strategy.

(a) Construct a conformal equivalence from U_s to U_0.

(b) Construct a conformal equivalence from U_0 to $\{z \in D \mid \operatorname{Im} z > 0\}$.

(c) Construct a conformal equivalence from $\{z \in D \mid \operatorname{Im} z > 0\}$ to $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$.

(5) Let Ω be the set $\{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < \frac{\pi}{2} \text{ and } \operatorname{Im} z > 0\}$. Construct a conformal equivalence from Ω to D.