## COMPLEX ANALYSIS
## HOMEWORK 9

(1) The main purpose of this exercise is to understand the automorphism of $\mathbb{C}$, $\mathrm{Aut}(\mathbb{C})$. Namely, $f : \mathbb{C} \to \mathbb{C}$ is bijective and analytic, and $f^{-1}$ is also analytic.

  (a) Suppose that $g(z)$ is an entire function function such that $\infty$ is *not* an essential singularity. That is to say, $w = 0$ is not an essential singularity of $g(1/w)$. Show that $g(z)$ must be a polynomial.

  (b) Let $f \in \mathrm{Aut}(\mathbb{C})$, prove that $\infty$ cannot be an essential singularity of $f$.

  (c) Characterize $\mathrm{Aut}(\mathbb{C})$.

(2) The main purpose of this exercise is to understand the automorphism of the Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

  (a) For any $T \in \mathrm{GL}(2; \mathbb{C})$ (invertible $2 \times 2$ matrices), it is not hard to see that its Möbius transform belong to $\mathrm{Aut}(\hat{\mathbb{C}})$.

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad \mu(T)(z) = \frac{az + b}{cz + d} \ .$$

  Check that $\mu(TT') = \mu(T) \circ \mu(T')$. It follows that $\mu(T^{-1})$ is the inverse map of $\mu(T)$.

  (b) Show that $\mu(T) = \mu(T')$ if and only if $T = sT'$ for some $s \in \mathbb{C}$. Therefore, $\mathrm{PGL}(2; \mathbb{C}) = \mathrm{GL}(2; \mathbb{C})/\{s\mathbf{I}\}$ is a subgroup of $\mathrm{Aut}(\hat{\mathbb{C}})$. Here, $\mathbf{I}$ is the $2 \times 2$ identity matrix.

  (c) For any $p \in \mathbb{C}$, show that there exists $T \in \mathrm{GL}(2; \mathbb{C})$ such that $\mu(T)(p) = \infty$.

  (d) Use Part (c) and Exercise (1) to conclude that $\mathrm{Aut}(\hat{\mathbb{C}}) = \mathrm{PGL}(2; \mathbb{C})$.

  (e) Given any three distinct points $z_0, z_1, z_2 \in \hat{\mathbb{C}}$, prove that there exists a unique element in $f \in \mathrm{Aut}(\hat{\mathbb{C}})$ such that $f(0) = z_0$, $f(1) = z_1$ and $f(\infty) = z_2$.

(3) Given a function $F : \hat{\mathbb{C}} \to \mathbb{C}$, define two functions on $\mathbb{C}$ as follows

$$\begin{aligned} f_0(z) &= F(z) \ , \\ f_1(w) &= F(1/w) \ \text{ for } w \neq 0 \quad \text{and} \quad f_1(0) = F(\infty) \ . \end{aligned}$$

  (a) A function $F : \hat{\mathbb{C}} \to \mathbb{C}$ is said to be holomorphic if both $f_0$ and $f_1$ are analytic functions on $\mathbb{C}$. Prove that a holomorphic function $F$ must be a constant.

  (b) A function $F : \hat{\mathbb{C}} \to \mathbb{C}$ (with isolated singularities) is said to meromorphic if both $f_0$ and $f_1$ are meromorphic functions on $\mathbb{C}$. A point $z \in \mathbb{C} \subset \hat{\mathbb{C}}$ is said to be a zero (or a pole) of $F$ if it is a zero (or a pole) of $f_1$, and the order is defined to be that of $f_0$. For $\infty \in \hat{\mathbb{C}}$, it is a zero (or a pole) of $F$ if 0 is a zero (or a pole) of $f_1$. Its order is defined by the same way.

Let $F : \hat{\mathbb{C}} \to \mathbb{C}$ be a meromorphic function. Given any $p \in \hat{\mathbb{C}}$, we say

$$(F) + p \geq 0$$

if $F$ is holomorphic, or holomorphic except at $p$ which is a pole of $F$ of order 1. For any fixed $p \in \hat{\mathbb{C}}$, characterize all such $F$.

(c) Discuss the general case of Part (b). Namely, given any finite number of points $\{p_n\}_{n=1}^N \in \hat{\mathbb{C}}$, define $(F) + \sum_{n=1}^N p_n$ analogously, and characterize those functions. Note that $p_n$'s could be the same.

(d) For a meromophic function $F : \hat{\mathbb{C}} \to \mathbb{C}$, what can you say about the total number of zeros and the total number of poles?