COMPLEX ANALYSIS
HOMEWORK 8

DUE: TUESDAY, DECEMBER 1

In what follows, Ω is an open and connected subset of $\mathbb{C}$, and $\mathcal{H}(\Omega)$ is the space of analytic function on Ω. In what follows, the convergence of functions means that they converge uniformly on any compact subset of Ω.

(1) Let D be the unit disk, $\{z \in \mathbb{C} \mid |z| < 1\}$. Show that a sequence $\{f_n\} \subset \mathcal{H}(D)$ converges to f if and only if $\int_{|z|=\rho} |f(z) - f_n(z)|\,|\mathrm{d}z| \to 0$ as $n \to \infty$ for $0 < \rho < 1$.

(2) Let $\{f_n\} \subset \mathcal{H}(\Omega)$ be a sequence of *injective* functions which converges to $f \in \mathcal{H}(\Omega)$. Show that f is either injective or a constant function.

(3) Suppose that $\{f_n\} \subset \mathcal{H}(\Omega)$ converges to a non-constant $f \in \mathcal{H}(\Omega)$. Let $z \in \Omega$ and $\alpha = f(z)$. Show that there is a sequence $\{z_n\} \in \Omega$ such that

$$\lim_{n\to\infty} z_n = z \qquad \text{and} \qquad f_n(z_n) = \alpha \quad \text{for } n \text{ being sufficiently large .}$$

(4) (a) Let f be an analytic function defined on some open set containing $\overline{B(z_0;\rho)}$. Show that

$$|f(z_0)|^2 \leq \frac{1}{\pi\rho^2} \int_0^{2\pi} \int_0^{\rho} |f(z_0 + re^{i\theta})|^2 \, r\mathrm{d}r\mathrm{d}\theta \ .$$

(b) Given any $M > 0$, let $\mathcal{F}_M = \{f \in \mathcal{H}(\Omega) \mid \iint_\Omega |f(z)|^2 \,\mathrm{d}x\mathrm{d}y \leq M\}$. Prove that $\mathcal{F}_M$ is normal.

(5) Given an entire function f, consider $\mathcal{F} = \{f(kz) \mid k \text{ is a constant}\}$. Let Ω be an annulus $\{z \in \mathbb{C} \mid \rho_1 < |z| < \rho_2\}$, and regard $\mathcal{F}$ as a subset of $\mathcal{M}(\Omega) \subset \mathcal{C}(\Omega; \hat{\mathbb{C}})$ (by restriction). Show that $\mathcal{F} \subset \mathcal{M}(\Omega)$ is normal if and only if f is a polynomial. Here, $\mathcal{M}(\Omega)$ is the space of all meromorphic functions on Ω. The convergence here is measured by the norm of the stereographic projection.