COMPLEX ANALYSIS
HOMEWORK 7

DUE: TUESDAY, NOVEMBER 3

(1) [p.200, #1] Prove the formula of Gauss:
$$2\pi\,\Gamma(z) = 3^{z-\frac{1}{2}}\,\Gamma(\frac{z}{3})\,\Gamma(\frac{z+1}{3})\,\Gamma(\frac{z+2}{3})\ .$$

(2) The Gauss psi function is defined by
$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}\ .$$

(a) Show that Ψ is meromoprhic on $\mathbb{C}$ with simple poles at $z = 0, -1, -2, \ldots$ with residue to be -1.

(b) Show that $\Psi(1) = -\gamma$.

(c) Show that $\Psi(z+1) - \Psi(z) = \frac{1}{z}$.

(d) Show that $\Psi(z) - \Psi(1-z) = -\pi\ \cot \pi z$.

(3) Evaluate $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ by the method of residues. You may consider the complex line integral of $\dfrac{1}{z^2}\dfrac{\pi\ \cos \pi z}{\sin \pi z}$ over the boundary of rectangle with vertices $\pm(n+\frac{1}{2}) \pm iY$.

(4) Read p.160 of Stein and Shakarchi. Show that
$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(z+n)} + \int_1^{\infty} e^{-t} t^{z-1} \mathrm{d}t\ .$$
for $z \neq 0, -1, -2, \ldots$, and not only for $\mathrm{Re}\, z > 0$ alone. You can compare it with the theorem of Mittag-Leffler.

(5) Let $f(z)$ be a non-trivial entire function with $f(0) = 1$,
$$M(\rho) = \sup_{|z|=\rho} |f(z)|\ ,\ n(\rho) = \#\{\text{zeros of } f(z) \text{ in } B(0;\rho),\ \text{counting multiplicity}\}\ .$$

(a) Show that
$$n(\rho) \leq \frac{\log M(2\rho)}{\log 2}\ .$$

(b) Evaluate
$$\int_0^{\rho} \frac{n(t)}{t} \mathrm{d}t\ .$$
It is related to the zeros of $f(z)$ in $B(0;\rho)$, and also $f(\rho\, e^{i\theta})$.