COMPLEX ANALYSIS
HOMEWORK 6

DUE: TUESDAY, OCTOBER 27

(1) [p.190, #1] Comparing coefficients in the Laurent developments of $\cot \pi z$ and of its expression as a sum of partial fractions, find the values of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\ , \qquad \sum_{n=1}^{\infty} \frac{1}{n^4}\ , \qquad \sum_{n=1}^{\infty} \frac{1}{n^6}\ .$$

(2) [p.193, #2] Prove that for $|z| < 1$

$$\prod_{k=0}^{\infty}(1 + z^{2^k}) = (1 + z)(1 + z^2)(1 + z^4)\cdots(1 + z^{2^k})\cdots = \frac{1}{1 - z}\ .$$

What you have to do is to estimate $\dfrac{1}{1-z} - \displaystyle\prod_{k=0}^{N}(1 + z^{2^k})$.

(3) [p.197, #1]: Suppose that $a_n \to \infty$ and are pairwise distinct, and A_n are nonzero complex numbers. Show that there exists an entire function with $f(a_n) = A_n$.

If you use the hint of Ahlfors, you have to prove the absolute convergence for $|z| \le R$. It ends up with estimating

$$\sum_{|a_n|>2R} \frac{|A_n|}{|g'(a_n)|}\Big|\exp\big(\gamma_n(z - a_n)\big)\Big| \qquad \text{for} \quad |z| \le R\ .$$

(4) [p.197, #2]: Prove that

$$\sin \pi(z + \alpha) = e^{\pi z \cot \pi\alpha} \prod_{n=-\infty}^{\infty}\left(1 + \frac{z}{n + \alpha}\right) e^{-\frac{z}{n+\alpha}}$$

where $\alpha \notin \mathbb{Z}$.

There is a hint in Ahlfors. Also, the formula on the right hand side might *not* be right.

(5) [p.198, #3]: Explain why $\cos\sqrt{z}$ is an entire function, and find out its genus.

(6) [p.206, #3]: With the fact that

$$\int_0^{\infty} e^{-t^2}\,\mathrm{d}t = \frac{1}{2}\sqrt{\pi}\ ,$$

apply the Cauchy theorem to evaluate the Fresnel integral $\displaystyle\int_0^{\infty}\sin(x^2)\,\mathrm{d}x$.

The hint is that the square of $(1 + i)/\sqrt{2}$ is i.