COMPLEX ANALYSIS
HOMEWORK 4

DUE: TUESDAY, OCTOBER 13

(1) Each of the following functions f has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity, define $f(0)$ so that f is analytic at $z = 0$; if it is a pole, find the singular part.

(a) $f(z) = \dfrac{\sin z}{z}$;

(b) $f(z) = \dfrac{\cos z}{z}$;

(c) $f(z) = \dfrac{\cos z - 1}{z}$;

(d) $f(z) = \dfrac{z^2 + 1}{z(z-1)}$;

(e) $f(z) = \dfrac{\cos(z^{-1})}{z^{-1}}$.

(2) Give the Laurent series expansion of

$$f(z) = \frac{1}{(z^2+1)(z^2-4)}$$

on the annulus of radius between 1 and 2. (You may start with its partial fraction.)

(3) [p.186, #4]: You are asked to determine its singular part, and to show that there is no z^{2k}-components in the Taylor series of its regular part, and then to calculate the first three coefficients.

(4) The following exercise is based on [p.136, #1∼#5]. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk. For any $a \in D$, let

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}\ .$$

You can check easily that φ_a defines an automorphism[1] of D. In fact, the inverse of φ_a is φ_{-a}.

- Check that $\varphi_a(\partial D) = \partial D$.
- Suppose that $|f(z)| \leq 1$ for $|z| \leq 1$. Prove that

$$|f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2}$$

[1]Namely, it is analytic and bijective, and the inverse map is also analytic.

for any $a \in D$. When will the equality hold? (You may consider $g = \varphi_{f(a)} \circ f \circ \varphi_{-a}$.)

- Suppose that f is an automorphism of D, and $f(a) = 0$. Show that $f = c\,\varphi_a$ for some constant c with $|c| = 1$. (Let h be the inverse of f. Compute the derivative of $h \circ f = z$ at a.)

You can think about what happens for the upper half plane, $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.

(5) Does there exist an analytic function $f : D \to D$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$?

(6) Is there an analytic function $f : D \to D$ with $f(0) = \frac{1}{2}$ and $f'(0) = \frac{3}{4}$? If so, find such an f. Is it unique?