**COMPLEX ANALYSIS**
**FINAL**

THURSDAY, JANUARY 14

[Total: 37 points] In what follows, $\Omega$ is always assumed to be an *open* and *connected proper* subset of $\mathbb{C}$.

(1) [7 points] True/False questions, no justifications needed.

(a) Given any three distinct complex numbers of unit length, $\zeta_1, \zeta_2$ and $\zeta_3$, define

$$f(z) = \int_0^z \frac{1}{\big((w-\zeta_1)(w-\zeta_2)(w-\zeta_3)\big)^{\frac{2}{3}}} \mathrm{d}w$$

as a function on the open unit disk. (Assume that the branch of 2/3-power is suitably chosen.) Then, we can find $\zeta_1, \zeta_2$ and $\zeta_3$ such that the image of $f(z)$ is a right triangle.

$\boxed{\text{F}}$ This is exactly the Schwarz–Christoffel formula. The image always has outer angle $2\pi/3$. In other words, the image is an equilateral triangle, and is never a right triangle.

(b) Suppose that $f(z)$ is a function on $\partial\Omega$, which is non-negative but not necessarily continuous. Let $u(z)$ be the harmonic function on $\Omega$ produced from the Perron's method with $f(z)$. Then, $u(z) \geq 0$ for any $z \in \Omega$.

$\boxed{\text{T}}$ Recall that $\mathcal{B} = \{v(z) : \text{subharmonic on } \Omega \mid \limsup_{z\to\zeta} v(z) \leq f(\zeta) \text{ for any } \zeta \in \partial\Omega\}$, and $u(z) = \sup_{v\in\mathcal{B}} v(z)$. It is clear that $0 \in \mathcal{B}$. Thus, $u(z) \geq 0$.

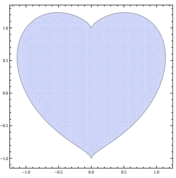
(c) Suppose that $\partial\Omega$ is compact, and $f(z)$ is a continuous function on $\partial\Omega$. Then, there exists a harmonic function $u(z)$ on $\Omega$, which extends continuously to $\partial\Omega$ and which is equal to $f(z)$ on $\partial\Omega$.

$\boxed{\text{F}}$ Consider $\Omega = D\backslash\{0\}$ with $f(z) = 0$ for $z \in \partial D$ and $f(0) = 1$. See Homework 12 #3b.

(d) For any $\alpha < -1$, there exists an automorphism of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $f(z)$, such that $f(0) = 0$, $f(1) = 1$ and $f(-1) = \alpha$.

$\boxed{\text{T}}$ See Homework 9 #2e.

(e) Let $\Omega$ be the region enclosed by the heart curve, $\Omega = \{x+iy \in \mathbb{C} \mid (x^2+y^2-1)^3 - x^2y^3 < 0\}$. Suppose that $\psi(z)$ is an automorphism of $\Omega$ with $\psi(0) = 0$. Then, $\psi(z)$ must be the identity map.



$\boxed{\text{F}}$ By the Riemann mapping theorem, $\Omega$ is conformal equivalent to $D$. Moreover, we may choose the map such that 0 is sent ot 0. But $z \mapsto e^{i\theta}z$ is an automorphism of $D$ fixing 0.

(f) Consider $\mathcal{F} = \{f_n(z) = \frac{1}{z+n}\}_{n\in\mathbb{N}}$ on $\mathbb{C}$. It is normal, in the sense of $\hat{\mathbb{C}}$.

$\boxed{\text{T}}$ We compute

$$\rho(f_n) = \rho(\frac{1}{f_n}) = \frac{2}{1+|z+n|^2} \leq 2 \; .$$

By Marty's theorem, it is normal. One can also conclude it directly.

(g) There exists a non-constant entire function $f(z)$, whose image does not contain the negative real axis.

$\boxed{\text{F}}$ One can apply little Picard theorem. Or compose a conformal map so that the image lie in the unit disk, and apply the Liouville's theorem.

(2) [5 points] Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Hint: The zeros of $\sin(\pi z)$ are exactly $\mathbb{Z}$. Since

$$\begin{aligned} \sin(\pi z) &= \sin(\pi x)\cosh(\pi y) + i\cos(\pi x)\sinh(\pi y) \quad \text{and} \\ \cos(\pi z) &= \cos(\pi x)\cosh(\pi y) - i\sin(\pi x)\sinh(\pi y) \; , \end{aligned}$$

it is not hard to show that

$$\frac{1}{2}|\sin(\pi z)| \leq |\cos(\pi z)| \leq 2|\sin(\pi z)|$$

when $|y| \geq 1000$. Another formula you might need is that

$$\int_{-\infty}^{\infty} \frac{1}{a^2+s^2}\mathrm{d}s = \frac{\pi}{a} \; .$$

Due to the argument principle,

$$\frac{1}{2\pi i}\int_{\gamma} g(z)\frac{f'(z)}{f(z)}\mathrm{d}z = \sum_{w_j\text{: zero of } f(z)} g(w_j)\, n(\gamma; w_j)$$

for any analytic function $f(z)$ and $g(z)$. Consider $g(z) = \frac{1}{z^2}$ and $f(z) = \sin(\pi z)$. Let $\gamma_{n,Y}$ be the boundary of the rectangle with vertices $\pm(\frac{1}{2}+n) \pm iY$. Since $g(z)$ is not analytic at

0, the above formula shall be corrected by the residue, $\mathrm{Res}(\frac{1}{z^2}\frac{\pi\cos(\pi z)}{\sin(\pi z)}, 0) = -\frac{\pi^2}{3}$. Then, the formula reads

$$\frac{1}{2\pi i}\int_{\gamma_{n,Y}} \frac{1}{z^2}\frac{\pi\cos(\pi z)}{\sin(\pi z)}\mathrm{d}z = 2\sum_{k=1}^{n}\frac{1}{k^2} - \frac{\pi^2}{3} \ .$$

We first consider the limit of the integral as $Y \to \infty$. For the horizontal segments of $\gamma_{n,Y}$, the integral is bounded by

$$\begin{aligned} C_1\int_{-(\frac{1}{2}+n)+iY}^{(\frac{1}{2}+n)+iY}\frac{1}{|z|^2}|\mathrm{d}z| &\leq C_2\int_{-\infty}^{\infty}\frac{1}{s^2+Y^2}\mathrm{d}s \\ &= \frac{C_2\pi}{Y} \xrightarrow{Y\to\infty} 0 \ . \end{aligned}$$

For the vertical segments of $\gamma_{n,Y}$, the integral is bounded by

$$\begin{aligned} C_3\int_{(\frac{1}{2}+n)-iY}^{(\frac{1}{2}+n)+iY}\frac{1}{|z|^2}\frac{|\sinh(\pi y)|}{|\cosh(\pi y)|}|\mathrm{d}z| &\leq C_3\int_{(\frac{1}{2}+n)-iY}^{(\frac{1}{2}+n)+iY}\frac{1}{|z|^2}|\mathrm{d}z| \\ &= C_3\int_{-\infty}^{\infty}\frac{1}{(n+\frac{1}{2})^2+s^2}\mathrm{d}s \\ &= \frac{C_3\pi}{n+\frac{1}{2}} \ . \end{aligned}$$

It follows that

$$\left|2\sum_{k=1}^{n}\frac{1}{k^2} - \frac{\pi^2}{3}\right| \leq \frac{C_3\pi}{n+\frac{1}{2}} \ .$$

By letting $n\to\infty$, we find that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

(3) [2 + 5 points] Suppose that $\Omega$ is simply-connected. Fix a point $z_0 \in \Omega$. Let

$$\mathcal{F} = \{f(z) : \text{injective and analytic on } \Omega \mid f(z_0) = 0, f'(z_0) > 0, |f(z)| < 1 \text{ for any } z \in \Omega\} \ .$$

(a) Explain that $\mathcal{F}$ is a normal family, in the sense of $\mathbb{C}$.

Since $|f(z)| < 1$, Montel's theorem says that $\mathcal{F}$ is normal.

(b) Suppose that there exists a $g(z) \in \mathcal{F}$ such that $g'(z_0) \geq f'(z_0)$ for any $f \in \mathcal{F}$. Prove that the image of $g(z)$ is the (open) unit disk. Namely, prove the surjectivity part of the Riemann mapping theorem.

Hint: Given $a \in D$, let $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$. Then, $\varphi_a'(0) = 1 - |a|^2$ and $\varphi_a'(a) = \frac{1}{1-|a|^2}$. Another fact is that $\frac{1+|a|}{2\sqrt{|a|}} > 1$ for any $a \in D\backslash\{0\}$.

Suppose that $w_0 \in D$ does not belong to the image of $g(z)$. Consider

$$h_1(z) = \frac{g(z) - w_0}{1 - \bar{w}_0 g(z)} \ ;$$

it is injective and the image is still contained in $D$. Since $h_1(z)$ misses zero and $\Omega$ is simply-connected, we can consider

$$h_2(z) = \sqrt{h_1(z)}\ .$$

Since $h_1(z) = (h_2(z))^2$, $h_2(z)$ is also injective. Finally, we adjust the map so that it sends $z_0$ to 0, and the derivative is positive at $z_0$. Namely, consider

$$h_3(z) = e^{i\theta}\frac{h_2(z) - h_2(z_0)}{1 - \overline{h_2(z_0)}\, h_2(z)}\ .$$

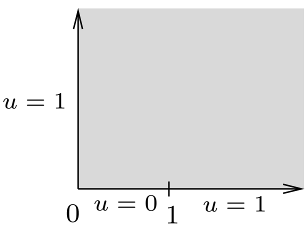
This function $f_3(z)$ also belongs to $\mathcal{F}$. A direct computation shows that

$$\begin{aligned} h_3'(z_0) &= e^{i\theta}\frac{1}{1-|h_2(z_0)|^2}\frac{1}{2}\frac{1}{\sqrt{h_1(z_0)}}(1-|w_0|^2)g'(z_0) \\ &= \frac{1-|w_0|^2}{2(1-|w_0|)\sqrt{|w_0|}}g'(z_0) \\ &= \frac{1+|w_0|}{2\sqrt{|w_0|}}g'(z_0) > g'(z_0)\ . \end{aligned}$$

This contradicts with the assumption that $g'(z_0)$ achieves the maximum among $\mathcal{F}$.

(4) [5 points] Construct a harmonic function $u(z)$ on the first quadrant, $\{z = x + iy \in \mathbb{C} \mid x > 0 \text{ and } y > 0\}$ with the following property:

- $u(z)$ extends continuously to the boundary except at the points 0 and 1;
- $u(z) = 1$ on the half-lines $\{y = 0, x > 1\}$ and $\{x = 0, y > 0\}$;
- $u(z) = 0$ on the segment $\{0 < x < 1, y = 0\}$.




Hint: On the upper half space, $\frac{1}{\pi}\arg(z)$ is harmonic, and extends continuously to the boundary except at the origin. Its value is 0 on the positive real axis, and is 1 on the negative real axis.

Let $\Omega$ be the first quadrant, and $\mathbb{H}$ be the upper half space. Consider

$$\Omega \xrightarrow{z^2} \mathbb{H} \xrightarrow{\frac{-z}{z-1}} \mathbb{H}\ .$$

For the boundary point, $0 \mapsto 0 \mapsto 0$, $1 \mapsto 1 \mapsto \infty$ and the segment $\{0 < x < 1, y = 0\}$ is sent to the positive real axis. Hence,

$$u(z) = \frac{1}{\pi}\arg\left(\frac{-z^2}{z^2-1}\right)$$

satisfies the desired property.

(5) [6 points] Let

$$\mathcal{F} = \{f(z) : \text{injective and analytic on } \Omega \mid f(z) \neq 0 \text{ for any } z \in \Omega\} \ .$$

Prove that $\mathcal{F}$ is a normal family, in the sense of $\hat{\mathbb{C}}$.

Suppose $\mathcal{F}$ is not normal. Apply the Zalcman lemma, and consider the rescaled limit $g(z)$: meromorphic on $\mathbb{C}$ with $\rho(g)(0) = 1$ and $\rho(g) \leq 1$.

Since $\mathcal{F}$ consists of analytic functions and $g(z)$ is not a constant function, $g(z)$ is analytic. In other words, $g(z)$ is an entire function. It is based on the argument principle.

We claim that $g(z)$ is also injective. If there are $z_1 \neq z_2$ such that $g(z_1) = g(z_2)$. We can consider

$$\frac{1}{2\pi i}\int_{\partial B(z_j,\epsilon)} \frac{g'(w)}{g(w) - g(z_j)}\mathrm{d}w = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\partial B(z_j,\epsilon)} \frac{\tilde{f}_n'(w)}{\tilde{f}_n(w) - g(z_j)}\mathrm{d}w$$

for $j = 1, 2$. By the argument principle, it violates the property that $f_n(z)$ is injective.

It follows that $g(z)$ is an injective entire function, which must be a degree one polynomial. But the similar argument shows that $g(z)$ must omit zero, and thus cannot be a polynomial. This is a contradiction.

**Remark.** By the Great Picard theorem, an injective entire function cannot have an essential singularity at $\infty$. Thus, $\infty$ is at most a pole of $g(z)$. It follows that $g(z)$ must be a polynomial. Since $g(z)$ is injective, it must have degree 1. This argument is almost the same as that for (6).

(6) [$3 + 4$ points] Let $f(z)$ be an entire function.

(a) Suppose that 0 is a pole of $g(w) = f(1/w)$. Show that $f(z)$ must be a polynomial.

Since 0 is pole of $g(w)$, there exist $n \in \mathbb{N}$ and $C > 0$ such that $|w|^n\,|g(w)| \leq C$ for any $|w| \leq 1$. Thus, $|f(z)| \leq C|z|^n$ for any $|z| \geq 1$. By Cauchy integral formula, it is not hard to show that $f^k(z) \equiv 0$ for any $k > n$. It follows that $f(z)$ is a polynomial.

(b) Suppose that $f(z)$ is not a polynomial. Then, given any $w_0 \in \mathbb{C}$, how many roots does the equation $f(z) = \zeta_0$ have? Explain your reason.

Hint: You can think about what happens for $f(z) = \exp(z)$.

We first recall the Great Picard theorem: a meromorphic function on the punctured disk $\{z \in \mathbb{C} \mid 0 < |z - z_0| < \delta\}$ that omits three values extend meromorphically to $z = z_0$.

Since $f(z)$ is not a polynomial and $g(w) = f(1/w)$ already omits $\infty$, $g(w)$ can only omits one value on any punctured neighborhood of $w = 0$. As a consequence, $f(z)$ can omit at most one value. Moreover, if $f(z)$ omits one value, all the other values must admit a preimage on any (punctured) small neighborhood of $w = 0$ (or $z = \infty$).

It follows that there are infinitely many preimages. Thus, $f(z) = \zeta_0$ admit infinitely many solutions with possible one exception of $\zeta_0$.