DIFFERENTIAL TOPOLOGY
HOMEWORK 13

DUE: MONDAY, MAY 26

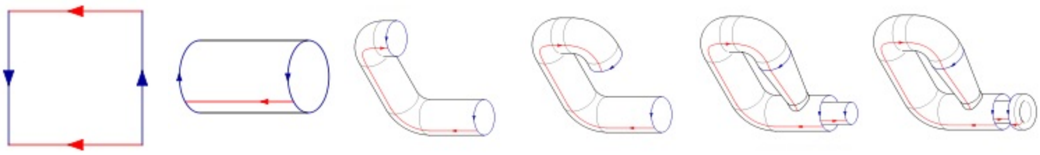
(1) (Inverse element of π_n) Let X be a topological space, and p be a point in X. Suppose that f is a continuous maps from $(I^n, \partial I^n)$ to (X, p). Define

$$\tilde{f}(s_1, s_2, \ldots, s_n) = f(1 - s_1, s_2, \ldots, s_n) \ .$$

Show that $f * \tilde{f}$ represents the trivial element in $\pi_n(X, p)$.

The following *pasting lemma* is useful. Suppose that $Y = P \cup Q$ where P and Q are both closed. Then, a map $g : Y \to X$ is continuous if $g|_P$ and $g|_Q$ are continuous.

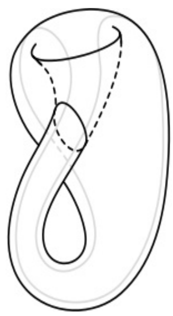
(2) (Klein bottle) Thom's theorem asserts that there is only one non-trivial class of $\mathcal{N}^2$, which is given by $\mathbb{RP}^2$. Explain that the Klein bottle represents a trivial element in $\mathcal{N}^2$, i.e. it bounds a compact 3-manifold.



The following description of the Klein bottle can help you. Let X be a manifold, and τ be a self-diffeomorphism of X. The *mapping torus* of (X, τ) is defined to be

$$\frac{I \times X}{(1, x) \sim (0, \tau(x))} \ .$$

Or equivalently, take $U_- = (-\epsilon, 1/2) \times X$, $U_0 = (0, 1) \times X$ and $U_+ = (1/2, 1 + \epsilon) \times X$. Identify $(-\epsilon, \epsilon) \times X \subset U_-$ and $(1 - \epsilon, 1 + \epsilon) \times X \subset U_+$ by $(s, \tau(x)) \sim (1 + s, x)$. The identifications with U_0 are basically given by the identity map. The latter description shows that the mapping torus is still a manifold. Note that the mapping torus is a fiber bundle over $\mathbf{S}^1$ with fiber to be X.



With this understood, the Klein bottle is the mapping torus of $(\mathbf{S}^1, \iota)$ where ι is the antipodal map. (*Hint.* You can try to 'fill' the fibers and 'extend' the self-diffeomorphism to construct a compact 3-manifold with boundary.)