DIFFERENTIAL TOPOLOGY
HOMEWORK 12

DUE: MONDAY, MAY 19

(1) Here is the abstract definition for smooth manifolds. A topological space M is called an n-dimensional manifold if there is an open cover $\mathscr{U} = \{U_i\}_{i\in I}$ such that

- for each $i \in I$, there is a continuous map $\varphi_i : U_i \to \mathbb{R}^n$ which maps U_i *homeomorphically* to an open subset of $\mathbb{R}^n$; these datum (U_i, φ_i) are called *coordinate charts*;
- the transitions between coordinate charts are smooth, i.e.

$$\varphi_j\varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \tag{0.1}$$

  is *smooth*; note that the domain and target space are both open subsets of $\mathbb{R}^n$, and the smoothness makes sense.

For compact manifold, it is not hard to prove that this definition coincides with the definition as a submanifold of $\mathbb{R}^n$. You can find a proof in [Hirsch, Theorem 3.4 of §1.3].

Consider the space of all 2-planes in $\mathbb{R}^4$. The space is the Grassmannian manifold $\mathrm{G}_{2,2}$. As explained in class, it is the quotient space

$$\rho : \mathrm{GL}(2;\mathbb{R})\backslash\mathrm{M}(2,4;2) \to \mathrm{G}_{2,2}\ .$$

The topology of $\mathrm{G}_{2,2}$ is given by the quotient topology. Let $V_0 \subset \mathrm{M}(2,4;2)$ be the open set consisting of the matrices whose *first two* columns are linearly independent, and $V_1 \subset \mathrm{M}(2,4;2)$ be the open set consisting of the matrices whose *last two* columns are linearly independent. Denote their images under ρ by U_0 and U_1, respectively.

(a) During the lecture, we constructed a homeomorphism $\varphi_0 : U_0 \to \mathbb{R}^4$. You can similarly construct a homeomorphism $\varphi_1 : U_1 \to \mathbb{R}^4$. Work out $\varphi_1\varphi_0^{-1}$, and check it is smooth.

The universal bundle γ_2^2 is defined by

$$\big\{([A], \vec{v}) \in \mathrm{G}_{2,2} \times \mathbb{R}^4 \,\big|\, \vec{v} \text{ belongs to the 2-plane spanned by the row vectors of } A\big\}\ .$$

(b) During the lecture, we constructed a trivialization $\pi^{-1}(U_0) \cong U_0 \times \mathbb{R}^2$. You can similarly construct a trivialization for $\pi^{-1}(U_1)$. Work out the transition map between $\pi^{-1}(U_0)$ and $\pi^{-1}(U_1)$. You shall get a smooth map from $U_0 \cap U_1 \to \mathrm{GL}(2;\mathbb{R})$.

(2) This exercise is a continuation of Exercise (3) and (4) of Homework 11. Consider the rank 2 subbundle L_k of $\Sigma \times \mathbb{R}^4$. What is the quotient bundle $\Sigma \times \mathbb{R}^4 / L_k$? Is it equivalent to L_k (or E_k) for some k? (*Hint.* Endow the fibers of $\Sigma \times \mathbb{R}^4$ with the standard inner product of $\mathbb{R}^4$. The quotient bundle is equivalent to the orthogonal bundle $L_k^{\perp}$.)