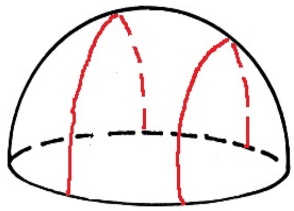
DIFFERENTIAL TOPOLOGY
HOMEWORK 11

DUE: MONDAY, MAY 12

(1) Explain that $\mathbb{RP}^2 = \mathbf{S}^2/\{\pm 1\}$ can be obtained by gluing a disk to the Möbius band. The following picture is the hint.



(2) Any element in $\mathrm{GL}(2;\mathbb{R})$ has a unique factorization as follows:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \rho & \frac{ac+bd}{\sqrt{a^2+b^2}} \\ 0 & \frac{ad-bc}{\rho} \end{bmatrix} \qquad (\spadesuit)$$

where $\rho > 0$ and $e^{i\phi}$ are defined by $(a, b) = (\rho\cos\phi, \rho\sin\phi)$. Note that $(a, b) \neq (0, 0)$.

Let M be a compact, connected, oriented surface whose boundary is diffeomorphic to $\mathbf{S}^1$. Suppose that $f : M \to \mathrm{GL}(2;\mathbb{R})$ is a smooth map. The main purpose of this exercise is to show that $\partial f : \mathbf{S}^1 \to \mathrm{GL}(2;\mathbb{R})$ is homotopic to a constant map. Moreover, the constant map can taken to be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if $\det(f) > 0$, and can taken to be $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ if $\det(f) < 0$.

According to ($\spadesuit$), f decomposes into two *smooth* maps, f_1 and f_2. The image of f_1 belongs to $\mathrm{SO}(2)$. The image of f_2 belongs to the space of non-degenerate upper-triangular matrices with positive $(1, 1)$-element.

(a) Show that ∂f_1 is homotopic to the constant map to the identity matrix. (*Hint.* You may invoke (3) of Homework 8.)

(b) Show that ∂f_2 is homotopic to a constant map. Moreover, the constant map can taken to be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if $\det(f) > 0$, and can taken to be $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ if $\det(f) < 0$.

(c) Combine Part (a) and (b) to show that ∂f is homotopic to a constant map with the desired property.

We remark that the same argument can be used to prove that $\mathrm{GL}(2;\mathbb{R})$ has two components.

(3) Let Σ be an oriented surface without boundary (also being compact and connected). Let o be a point on Σ, and U be a coordinate neighborhood of o. Assume U is diffeomorphic to $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<2\}$. Denote the complement of o by V. It is clear that U and V constitute a locally finite open cover of Σ.

For any integer k, introduce the rank 2 vector bundle E_k by the following transition function

$$g^k_{U,V}(x,y) = \begin{bmatrix} r^k\cos(k\theta) & r^k\sin(k\theta) \\ -r^k\sin(k\theta) & r^k\cos(k\theta) \end{bmatrix} \in \mathcal{C}^\infty(U\cap V; \mathrm{GL}(2;\mathbb{R})) \qquad (\heartsuit)$$

where the $(x,y) = (r\cos\theta, r\sin\theta)$ is the coordinate on U.

(a) Suppose that k and ℓ are two *distinct*, *nonnegative* integers. Prove that E_k and E_ℓ are *not equivalent*. (*Hint.* Suppose they are equivalent. Namely, there exist $h_U \in \mathcal{C}^\infty(U;\mathrm{GL}(2;\mathbb{R}))$ and $h_V \in \mathcal{C}^\infty(V;\mathrm{GL}(2;\mathbb{R}))$ such that the following diagram commutes over $U\cap V$:

$$\begin{array}{ccc} V\times\mathbb{R}^2 & \xrightarrow{(p,h_V(p))} & V\times\mathbb{R}^2 \\ {\scriptstyle (p,g^k_{U,V}(p))}\big\downarrow & & \big\downarrow{\scriptstyle (p,g^\ell_{U,V}(p))} \\ U\times\mathbb{R}^2 & \xrightarrow{(p,h_U(p))} & U\times\mathbb{R}^2 \end{array}$$

In other words, we have $g^k_{U,V}(p) = (h_U(p))^{-1}\cdot g^\ell_{U,V}(p)\cdot h_V(p)$ for any $p\in U\cap V$. Use (2) to show that $g^k_{U,V}$ is homotopic to $g^\ell_{U,V}$ as maps from $U\cap V$ to $\mathrm{GL}(2;\mathbb{R})$. And try to get a contradiction.)

(b) Show that E_k and E_{-k} are equivalent.

Here are two remarks.

- By introducing the *orientation* for vector bundles, we can distinguish E_k from E_{-k}.
- A similar construction produces a rank d vector bundle on an n-dimensional manifold from a smooth map from $\mathbf{B}^n\backslash\{0\}$ to $\mathrm{GL}(d;\mathbb{R})$. The topology of the the resulting bundle is determined by the homotopy class of the map from $\mathbf{S}^{n-1}$ (or $\mathbf{B}^n\backslash\{0\}$) to $\mathrm{GL}(d;\mathbb{R})$.

(4) This exercise is a continuation of (3). Suppose that $\chi(r)$ is a smooth function which is equal to 1 when $r\le 1$ and vanishes when $r\ge\sqrt{2}$. For any integer k, define a rank 2 subbundle L_k of $\Sigma\times\mathbb{R}^4$ as follows:

- For $(x,y)\in U$, $L_k|_{(x,y)}$ consists of the vectors $(u_1,u_2,v_1,v_2)\in\mathbb{R}^4$ satisfying

$$\begin{bmatrix} r^k\cos(k\theta) & -r^k\sin(k\theta) \\ r^k\sin(k\theta) & r^k\cos(k\theta) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \chi(r)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \ .$$

- Outside U, the fiber of L_k is the subspace $\{(0,0,v_1,v_2)\}\subset\mathbb{R}^4$.

Prove that L_k is equivalent to the bundle E_k defined in (3).