DIFFERENTIAL TOPOLOGY
HOMEWORK 10

DUE: MONDAY, MAY 5

(1) Let (E, B, π, F) be a fiber bundle, where E, B and F are boundaryless manifolds. Suppose that both B and F are connected. Show that E must be connected. (*Remark.* For general topological spaces, being 'connected' may be different from being 'path-connected'. But these two notions coincide for manifolds.)

(2) Consider the Hopf map

$$\begin{array}{rccc} \pi : & \mathbf{S}^3 & \to & \mathbf{S}^2 \\ & (x_1, x_2, x_3, x_4) & \mapsto & (2\,\mathrm{Re}(z\bar{w}), 2\,\mathrm{Im}(z\bar{w}), |z|^2 - |w|^2) \end{array}$$

where $z = x_1 + ix_2$ and $w = x_3 + ix_4$.

(a) Check that the image of π does belong to $\mathbf{S}^2$.

(b) Show that π is a submersion, and $\pi^{-1}(y)$ is diffeomorphic to $\mathbf{S}^1$ for any $y \in \mathbf{S}^2$.

(c) Show that $(\mathbf{S}^3, \mathbf{S}^2, \pi, \mathbf{S}^1)$ constitutes a fiber bundle. (*Hint.* It suffices to show that $\pi^{-1}(\mathbf{S}^2 \backslash \{(0, 0, -1)\} \cong \mathbb{R}^2)$ is diffeomorphic to $\mathbb{R}^2 \times \mathbf{S}^1$ with the required property of a fibration. The key is to construct a map from $\pi^{-1}(\mathbf{S}^2 \backslash \{(0, 0, -1)\})$ to $\mathbf{S}^1$.)

(d) Calculate the Hopf invariant of this map π.

(e) Let $f_0 : \mathbf{S}^3 \to \mathbf{S}^2$ be a null-homotopic map (i.e. homotopic to a constant map). Calculate the Hopf invariant for f_0.

(3) Suppose that there are two maps: $f : \mathbf{S}^3 \to \mathbf{S}^2$ and $g : \mathbf{S}^2 \to \mathbf{S}^2$. Show that $H(g \circ f) = (\deg(g))^2 H(f)$