DIFFERENTIAL TOPOLOGY
HOMEWORK 8

DUE: MONDAY, APRIL 14

The following exercises are taken from [GP].

(1) Suppose that Z is a hypersurface in an oriented manifold Y, i.e. a submanifold of codimension 1. Assume both Z and Y are boundaryless for simplicity. Prove that Z is orientable if and only if there exists a *smooth normal vector field* $\vec{n}(z)$ along Z in Y. (*Hint.* From an oriented basis of T_zZ and T_zY, construct a unit-normed vector perpendicular to T_zZ.)

(2) Let $f(z) = 1/z$.
   (a) Consider the map $f/|f|$ from the circle of radius r to S^1. Compute $\deg(f/|f|)$.
   (b) Why does the proof of the fundamental theorem of algebra *not* imply that $1/z = 0$ for some $z \in \mathbb{C}$?

(3) For any map $f : S^1 \to S^1$, there exists a map $g : \mathbb{R}^1 \to \mathbb{R}^1$ such that
$$f(\cos t, \sin t) = (\cos(g(t)), \sin(g(t))) \ ,$$
and g obeys
$$g(t + 2\pi) = g(t) + 2\pi q$$
for some $q \in \mathbb{Z}$.
   (a) Check that $\deg(f) = q$. (*Hint.* According to the definition, degree is the signed count of the total number of points of the pre-image of a regular value.)
   (b) Prove that two maps from S^1 into itself are homotopic if they have the same degree. (*Remark.* We explained the 'only if' part in class.)
   (c) Prove that f can be extended to a map $F : B^2 \to S^1$ if $\deg(f) = 0$. (*Hint.* Let A be the annulus of radius between $\frac{1}{2}$ and 1. Use Part (b) to extend f over A such that the map on the inner circle is the constant map.)

(4) Let p be a Lefschetz fixed point of $f : X \to X$, and denote by $L_p(f)$ the local Lefschetz number of f at p.
   (a) Show that $f(x) = 2x$ on $\mathbb{R}^k$ has $L_0(f) = 1$.
   (b) Show that $f(x) = \frac{1}{2}x$ on $\mathbb{R}^k$ has $L_0(f) = (-1)^k$.
   (c) Prove that $\chi(S^k) = 2$ if k is even, and $\chi(S^k) = 0$ if k is odd. (*Hint.* Use Part (a) and (b).)