DIFFERENTIAL TOPOLOGY
HOMEWORK 7

DUE: MONDAY, APRIL 7

The following exercises are taken from [GP].

(1) Compute the orientation of $X \cap Z$ in the following examples by exhibiting positively oriented bases at every point. By convention, we orient $\mathbb{R}^3$ by (e_1, e_2, e_3). Orient xy-plane by (e_1, e_2), the yz-plane by (e_2, e_3). Finally, orient S^1 and S^2 as the boundary of B^2 and B^3, respectively.
   (a) $X = S^1$, $Z = y$-axis in $\mathbb{R}^2$.
   (b) $S = S^2$, $Z = yz$-plane in $\mathbb{R}^3$.
   (c) $X = S^1$ in the xy-plane, $Z = yz$-plane in $\mathbb{R}^3$.
   (d) $X = \{(x, y, z) \mid x^2 + y^2 - z^2 = 1\}$ with the orientation induced by $+1 \in \mathbb{R}^1$, $Z = xy$-plane in $\mathbb{R}^3$.

(2) Let $p(z)$ be a complex polynomial. Suppose that $\Omega \subset \mathbb{R}^2$ is a compact region whose boundary is a smooth curve. Show that if $p(z)$ has no zeros on $\partial\Omega$, then the total number of zeros of $p(z)$ inside Ω, counting multiplicities, is the degree of the map $p/|p| : \partial\Omega \to S^1$. (*Hint.* The zeros of a complex polynomial are discrete, and thus are finite in Ω. Denote the zeros by $\{z_j\}_{j=1}^n$. We may choose a small disk $D_j \subset \Omega$ around z_j such that these disks are disjoint from each other. Consider the map $p/|p| : W - \bigcup_{j=1}^n D_j \to S^1$.)

(3) (a) Let X be an orientable manifold. Show that both orientations on X induce the same product orientation on $X \times X$.

   (b) Let Z be a compact submanifold of Y, both oriented, with $\dim Z = \frac{1}{2} \dim Y$. We have explained in class that

$$I(Z, Z) = I(Z \times Z, \Delta) \tag{0.1}$$

   where Δ is the diagonal of Y. Since both hand sides are zero when $\dim Z$ is odd, the sign $(-1)^{\dim Z}$ is not needed here. Note that the left hand side is the intersection number in Y, and the right hand side is the intersection number in $Y \times Y$.

   Prove that the Euler characteristic of an orientable manifold X is independent of the choice of orientation. (*Hint.* Apply (0.1) for $Z = \Delta$ in $Y = X \times X$. In this case, all the manifolds appear in the right hand side of (0.1) is a square, and then invoke Part (a).)