DIFFERENTIAL TOPOLOGY
HOMEWORK 6

DUE: MONDAY, MARCH 31

The following exercises are taken from [GP].

(1) (a) If $f : X \to Y$ is homotopic to a constant map, show that $I_2(f, Z) = 0$ for all complementary dimensional closed $Z \subset Y$, except perhaps if $\dim X = 0$. (*Hint.* Show that if $\dim Z < \dim Y$, then f can be homotopic to a constant map $X \to \{y\}$ with $y \notin Z$.)

(b) If X is one point, for which Z will $I_2(f, Z) \neq 0$?

(2) A manifold is said to be *contractible* if the identity map is homotopic to the constant map. For instance, Euclidean spaces are contractible.

(a) Prove that the intersection theory is vacuous for contractible manifolds. That is to say, if Y is contractible of positive dimension, then $I_2(f, Z) = 0$ for any $f : X \to Y$ where X is compact of positive dimension, Z is closed, and they are of complementary dimension.

(b) Show that except the one-point space, no compact (and connected) manifolds is contractible. (*Hint.* Apply part (a) for the identity map with Z to be any point in Y.)

(3) This exercise asks you to finish the missing step in the proof of the Jordan–Brouwer separation theorem. Suppose that X is a compact, connected hypersurface in $\mathbb{R}^n$. Given a point z in $\mathbb{R}^n \backslash X$ and a direction vector $\vec{v}$ in S^{n-1}, consider the ray $\ell(\vec{v})$ emanating from z in the direction of $\vec{v}$:

$$\ell(\vec{v}) = \left\{ z + t\vec{v} \in \mathbb{R}^n \,\middle|\, t > 0 \right\} .$$

(a) Check that $\ell(\vec{v})$ is transversal to X if and only if $\vec{v}$ is a regular value of the map

$$\begin{array}{rccc} u: & X & \to & S^{n-1} \\ & x & \mapsto & \dfrac{f(x) - z}{|f(x) - z|} \end{array}$$

where $f : X \to \mathbb{R}^n$ is the inclusion map.

(b) Prove that almost every $\vec{v} \in S^{n-1}$ achieves the transversality condition of (a). (*Hint.* This is a typical application of the parametric transversality theorem.)

(c) Suppose that $\vec{v}$ satisfies the transversality condition. Let z' be a point in $\ell(\vec{v})$ and $z' \notin X$. Show that $W_2(X, z) = W_2(X, z') + k \pmod 2$ where k is the number of times $\ell(\vec{v})$ intersects X between z and z'.

With this understood, it is not hard to show that there exists some $z' \notin X$ such that $W_2(X, z') = 1$. (You don't have to submit this part.)

(4) Let $F : S^n \to \mathbb{R}^n$ be a smooth function satisfying the symmetry condition $F(-x) = -F(x)$ for any $x \in S^n$. Show that there exits some $x' \in S^n$ such that $F(x') = 0$. (*Hint*. If not, consider that map $S^n \to S^n$ which maps x to $(\frac{F(x)}{|F(x)|}, 0)$. What can you say about its (mod 2) degree?)