DIFFERENTIAL TOPOLOGY
HOMEWORK 5

DUE: MONDAY, MARCH 24

The following exercises are taken from [GP]. In what follows, X, Y, Z are assumed to be *compact* manifolds *without boundary.*

(1) Let $f : X \to Y$ be a smooth map which is one-to-one on $Z \subset X$. Suppose that

$$df|_x : T_xX \to T_{f(x)}Y \tag{0.1}$$

is an isomorphism for all $x \in Z$. It follows that $f : Z \to Y$ is an embedding, and Z is diffeomorphic to $f(Z)$. Prove that f, in fact, maps an open neighborhood of Z diffeomorphically onto a neighborhood of $f(Z)$. (*Hint.* The condition (0.1) implies that f must be a local diffeomorphism on some neighborhood of Z. It remains to show that f is one-to-one on a (possibly smaller) neighborhood of Z. You can prove it by contradiction, and use the fact that f is one-to-one on Z.)

The statement works for *non-compact* Z as well. The key ingredient is that partition of unity guarantees a *locally finite* open cover. You can see [GP, Exercise 14 of §I.8] for more. (You don't have to submit the non-compact case.)

(2) Consider $Z \subset Y \subset \mathbb{R}^N$. The *normal bundle* of Z in Y is defined to be the set

$$N(Z;Y) = \big\{(z,v) \subset TY \,\big|\, z \in Z \text{ and } v \perp T_zZ\big\}$$

where the orthogonal complement of T_yZ in T_yY is taken with respect to the induced inner product from $\mathbb{R}^N$. It is not hard to check that $N(Z;Y)$ is a manifold of the same dimension as Y. You can see [GP, Proposition on P.71] and [GP, Exercise 12 of §II.3] for the detail. (You don't have to submit this part.)

Note that Z naturally embeds in $N(Z;Y)$ by letting $v = 0$. Prove that there exists a diffeomorphism from an open neighborhood of Z in $N(Z;Y)$ to a open neighborhood of Z in Y. (*Hint.* Let Y^ϵ be the ϵ-neighborhood of Y in $\mathbb{R}^N$. Consider the map $h : N(Z;Y) \to \mathbb{R}^N$ defined by $(z,v) \mapsto z + v$. The set $W = h^{-1}(Y^\epsilon)$ is an open neighborhood of Z in $N(Z;Y)$. Apply Exercise (1) for $W \xrightarrow{h} Y^\epsilon \xrightarrow{\pi} Y$.)

(3) (a) Let Δ be the diagonal in $X \times X$. Show that $N(\Delta; X \times X)$ is $\{(x,v,x,-v) \in T_xX \times T_xX \,|\, v \in T_xX\}$.

(b) It is not hard to see that $TX \to N(\Delta; X \times X)$ defined by $(x,v) \mapsto (x,v,x,-v)$ is a diffeomorphism. Prove that there is a diffeomorphism of a neighborhood of X in TX and a neighborhood of Δ in $X \times X$.

(4) Suppose that X and Z are two submanifolds in Y with $\dim X + \dim Z < \dim Y$. Prove that X may be pulled away from Z by an arbitrary small deformation: given $\epsilon > 0$, there exists a deformation $X_t = \iota_t(X)$ such that X_1 does not intersect Z and $|x - \iota_1(x)| < \epsilon$ for any $x \in X$. (*Note.* You have to make sure that ι_t is an embedding for all $t \in [0, 1]$. We prove the stability theorem for homotopy. Namely, the parameter space is $[0, 1]$. It is an easy exercise to show that the stability theorem still holds if the parameter space is an open ball in some Euclidean space. In fact, the proof is completely the same.)