DIFFERENTIAL TOPOLOGY
HOMEWORK 4

DUE: MONDAY, MARCH 17

The following exercises are taken from [GP]. You only have to choose 6 out of 7, but you cannot skip (3). If you do all of them, you will get some bonus points.

(1) Consider the unit sphere S^3 in $\mathbb{R}^4$. Show that TS^3 is diffeomorphic to $S^3 \times \mathbb{R}^3$. (*Hint.* Any $(x, y, z, w) \in S^3$ is always perpendicular to the following three vectors: $(-y, x, -w, z)$, $(-z, w, x, -y)$, $(-w, -z, y, x)$.)

(2) The *projection* is a canonical map from the tangent bundle to the original manifold. It is defined by

$$\begin{array}{rccl} \pi: & TX \subset \mathbb{R}^N \times \mathbb{R}^N & \to & X \subset \mathbb{R}^N \\ & (x, v) & \mapsto & x \end{array} .$$

Check that π is a submersion.

(3) A *vector field* $\vec{v}$ on a manifold $X \subset \mathbb{R}^N$ is a smooth map $\vec{v} : X \to \mathbb{R}^N$ such that $\vec{v}(x) \in T_x X$ for any $x \in X$. Equivalently, it is a smooth map from X to TX such that $\pi \circ \vec{v}$ is the identity map on X. For instance, the three vectors given in the hint for (1) are defined at every point of S^3, and thus are vector fields on S^3.

A point $x \in X$ is called a *zero* of the vector field $\vec{v}$ if $\vec{v}(x) = 0$. A vector field $\vec{v}$ is said to be *nowhere zero* if it has no zeros.

(a) If k is a positive odd integer, show that there exists a nowhere zero vector field $\vec{v}$ on S^k.

(b) Prove that if S^k has a nowhere zero vector field, then the antipodal map is homotopic to the identity map. (*Hint.* It suffices to show that both maps are homotopic to the map $\vec{v}/|\vec{v}|$).

(4) Prove the (*weak*) *Whitney immersion theorem*: every k-dimensional manifold can be immersed in $\mathbb{R}^{2k}$. (*Hint.* For an immersion $f : X \subset \mathbb{R}^N$, consider the derivative map $TX \to \mathbb{R}^N$ defined by $(x, v) \mapsto \mathrm{d}f|_x(v)$.)

(5) Show that any manifold X admits a proper map $\rho : X \to \mathbb{R}$.

(6) Let X be a k-dimensional manifold with boundary. Show that ∂X is a $(k-1)$-dimensional manifold without boundary.

(7) (a) Show that the fixed point in the Brouwer fixed-point theorem needs not to be an interior point.

(b) Show that the Brouwer fixed-point theorem is false for the open ball $\{x \in \mathbb{R}^n \,\big|\, |x| < 1\}$.