DIFFERENTIAL TOPOLOGY
HOMEWORK 3

DUE: MONDAY, MARCH 10

The following exercises are taken from [GP].

(1) (a) Suppose that $f_0, f_1 : X \to Y$ are homotopic. Show that there exists a homotopy $\tilde{F} : X \times I \to Y$ such that $\tilde{F}(x,t) = f_0(x)$ for all $t \in [0, \frac{1}{4}]$ and $\tilde{F}(x,t) = f_1(x)$ for all $t \in [\frac{3}{4}, 1]$. (*Hint.* Construct a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho(t) = 0$ if $t \in [0, \frac{1}{4}]$ and $\rho(t) = 1$ if $t \in [\frac{3}{4}, 1]$.)

(b) Prove that the homotopy is an equivalence relation. Namely, if $f \sim g$ and $g \sim h$, then $f \sim h$.

(2) If k is odd, show that the antipodal map

$$\begin{array}{rccc} \alpha : & S^k & \to & S^k \\ & x & \mapsto & -x \end{array}$$

is homotopic to the identity map.

(3) Let $\rho : \mathbb{R} \to \mathbb{R}$ be a function with $\rho(s) = 1$ if $|s| \leq 1$ and $\rho(s) = 0$ if $|s| \geq 2$. Define $f_t : \mathbb{R} \to \mathbb{R}$ by $f_t(x) = x\rho(tx)$. Verify that f_0 is a diffeomorphism, but f_t is not for any $t > 0$. Why can't we apply the stability theorem here?

(4) A *deformation* of a submanifold $Z \subset Y$ is a smooth homotopy $H : Z \times I \to Y$ such that $H(\,\cdot\,, 0)$ is the inclusion map of Z and each $H(\,\cdot\,, t)$ is an *embedding*. Thus, $\big\{Z_t = H(Z,t)\big\}_{t \in I}$ is a smoothly varying submanifold of Y with $Z_0 = Z$. Show that if Z is compact, then *any homotopy* F of its inclusion map is a deformation for sufficiently small t.

(5) (a) Prove that the rational numbers have measure zero in $\mathbb{R}$, even though they are dense.

(b) Exhibit a smooth function $f : \mathbb{R} \to \mathbb{R}$ whose set of critical values is dense. (*Hint.* Write the rational numbers in a sequence $r_0, r_1, \ldots$. Construct a smooth function on $[i, i+1]$ that is zero near the endpoints and that has r_i as a critical value.)

(6) (a) A manifold M is said to be *simply connected* if any map $f : S^1 \to M$ is homotopic to a constant map. Show that $\mathbb{R}^k$ is simply connected.

(b) Prove that S^k is simply connected for any $k > 1$. (*Hint.* Use Sard theorem and the stereographic projection.)