DIFFERENTIAL TOPOLOGY
HOMEWORK 2

DUE: MONDAY, MARCH 3

The following exercises are taken from [GP].

(1) (a) Let $x_1, x_2, \ldots, x_N$ be the standard coordinate functions on $\mathbb{R}^N$, and let X be a k-dimensional manifold in $\mathbb{R}^N$. Prove that every point $p \in X$ has a neighborhood on which the restrictions of some k-coordinate functions $x_{j_1}, x_{j_2}, \ldots, x_{j_k}$ form a local coordinate system. (*Hint.* Let $e_1, e_2, \ldots, e_N$ be the usual basis for $\mathbb{R}^N$. First, prove that there are k directions $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ such that the projection from $\mathbb{R}^N$ onto the subspace spanned by $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ is a linear isomorphism when restricted on T_pX. Then invoke the inverse function theorem.)

(b) For simplicity, assume that $x_1, x_2, \ldots, x_k$ form a local coordinate on a neighborhood V of $p \in X$. Show that there are $(N-k)$ smooth functions $g_{k+1}, g_{k+2}, \ldots, g_N$ on some open set U in $\mathbb{R}^k$ such that V may be taken to be the set

$$\left\{ (x_1, \ldots, x_k, g_{k+1}(x), \ldots, g_N(x)) \in \mathbb{R}^N \,\middle|\, x = (x_1, \ldots, x_k) \in U \right\} .$$

That is to say, every manifold can locally expressed as a graph.

(2) (a) If $f : X \to Y$ is a submersion and $U \subset X$ is an open subset, show that $f(U)$ is open in Y. (*Hint.* Suppose that $V \subset \mathbb{R}^k$ is open. Then, for any $p \in V$, there exists an open neighborhood of $p \in V$ which is a product of open intervals.)

(b) If X is compact and Y is connected, prove that every submersion $f : X \to Y$ is surjective. (*Hint.* It suffices to prove that $f(X)$ is both open and closed in Y.)

(c) Show that there exists no submersions of compact manifolds into Euclidean spaces.

(3) (a) Check that 0 is the only critical value of the map

$$\begin{array}{cccc} f: & \mathbb{R}^3 & \to & \mathbb{R}^1 \\ & (x,y,z) & \mapsto & x^2 + y^2 - z^2 \end{array} .$$

(b) Show that if a and b satisfy $ab > 0$, then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic.

(c) Draw its picture to understand the catastrophic change in the topology of $f^{-1}(c)$ as c passes from negative to positive values.

(4) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps of manifolds. Suppose that g is transversal to a submanifold W of Z. Prove that $f \pitchfork g^{-1}(W)$ if and only if $(g \circ f) \pitchfork W$.

(5) Let $V = \mathbb{R}^k$ be the k-dimensional Euclidean space, and let $\triangle$ be the diagonal of $V \times V$. For a linear map $A : V \to V$, consider the graph $\Gamma_A = \{(v, Av) \in V \times V \mid v \in V\}$. Show that $\Gamma_A \pitchfork \triangle$ if and only if $+1$ is not an eigenvalue of A.