DIFFERENTIAL TOPOLOGY
HOMEWORK 1

DUE: MONDAY, FEBRUARY 24

The following exercises are taken from [GP].

(1) Let $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}^M$, $Z \subset \mathbb{R}^L$ be arbitrary subsets. Let $f : X \to Y$ and $g : Y \to Z$ be smooth maps. Check that the composition map $g \circ f : X \to Z$ is smooth.

(2) Let $B_a = \left\{ \mathbf{x} \in \mathbb{R}^k \,\middle|\, |\mathbf{x}| < a \right\}$, where a is some positive number. Show that the map

$$\mathbf{x} \mapsto \frac{a\mathbf{x}}{\sqrt{a^2 - |\mathbf{x}|^2}}$$

is a diffeomorphism of B_a onto $\mathbb{R}^k$.

(3) Give a smooth map $f : \mathbb{R}^1 \to \mathbb{R}^1$ which is one-to-one and onto but is *not* a diffeomorphism.

(4) The *graph* of a smooth map $f : X \to Y$ is defined by

$$\mathrm{graph}(f) = \left\{ (x, f(x)) \subset X \times Y \,\middle|\, x \in X \right\} .$$

There is a natural smooth map $F : X \to \mathrm{graph}(f) \subset X \times Y$ defined by $F(x) = (x, f(x))$. Show that if f is smooth, then F is a diffeomorphism. Hence, $\mathrm{graph}(f)$ is a manifold if X is.

(5) Let $a > 0$ be a positive number.
 (a) Check that the hyperboloid in $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = a$ is a manifold.
 (b) Describe the tangent space to the hyperboloid $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$.
 (c) Why doesn't $x^2 + y^2 - z^2 = 0$ define a manifold?

(6) The purpose of this exercise is to construct a very useful function, the *cut-off* function.
 (a) Consider the function $\psi : \mathbb{R}^1 \to \mathbb{R}^1$ defined by

$$\psi(x) = \begin{cases} \exp(-\frac{1}{x^2}) & \text{if } x > 0 \ , \\ 0 & \text{if } x \leq 0 \ . \end{cases}$$

 Check that ψ is smooth.
 (b) Show that $\varphi(x) = \psi(x - a)\psi(b - x)$ is a smooth function, positive on (a, b) and zero elsewhere. (Here $a < b$.)

With the function φ,

$$\chi(x) = \frac{\int_{-\infty}^{x} \varphi(t)\mathrm{d}t}{\int_{-\infty}^{\infty} \varphi(t)\mathrm{d}t}$$

is a smooth function satisfying $\chi(x) = 0$ for $x \leq a$, $\chi(x) = 1$ for $x \geq b$, and $0 < \chi(x) < 1$ for $x \in (a, b)$.