DIFFERENTIAL GEOMETRY I
HOMEWORK 14

DUE: WEDNESDAY, DECEMBER 31

(1) Let $P \to M$ be a principal $\mathrm{O}(k)$-bundle. Let ρ be the standard representation of $\mathrm{O}(k)$ on $\mathbb{R}^k$, namely, $\rho : \mathrm{O}(k) \to \mathrm{Gl}(k;\mathbb{R})$ is the inclusion map. Prove that $E = P \times_\rho \mathbb{R}^k$ naturally carries a bundle metric whose orthonormal frame bundle[1] is exactly P. [*Hint*: It is always true locally.]

Remark 1: Different Lie group G corresponds to different (fiberwise) *geometric structure*. Here are some examples.

- $\mathrm{Gl}_+(k;\mathbb{R}) = \{\mathfrak{m} \in \mathrm{Gl}(k;\mathbb{R}) \mid \det \mathfrak{m} > 0\}$: orientation.
- $\mathrm{Sl}(k;\mathbb{R})$: fiberwise determinant (a nonwhere vanishing section of $\Lambda^k E^*$)
- $\mathrm{SO}(k)$: metric and fiberwise determinant.
- $\mathrm{Gl}(k;\mathbb{C})$: almost complex structure.
- $\mathrm{U}(k)$: Hermitian metric.
- $\mathrm{SU}(k)$: Hermitian metric and complex determinant[2].

The story is quite interesting for exceptional Lie groups.

Remark 2: You can compare this exercise with (4) and (5) of Homework 13.

(2) Let G be a *connected* Lie group. Prove that any principal G-bundle over $\mathbf{S}^1$ is isomorphic to the trivial bundle, $\mathbf{S}^1 \times G$. [*Hint*: It is true over any open arc of $\mathbf{S}^1$. Also, it suffices to construct a global section to show that a principal G-bundle is trivial.]

(3) Consider the 1-form $A = \mathrm{d}z + x\,\mathrm{d}y$ on $\mathbb{R}^3$. Since A is nowhere vanishing, $H_A = \ker A$ is a rank 2 subbundle of $T\mathbb{R}^3$. Is H_A involutive?

(4) Consider the trivial $\mathbb{C}^1$-bundle over $\mathbb{R}^2$, $\mathbb{R}^2 \times \mathbb{C}^1$ with the covariant derivative $\nabla = \mathrm{d} + i(x\,\mathrm{d}y - y\,\mathrm{d}x)$.

(a) Calculate the curvature F_∇.

(b) For any $\tau > 0$, the parallel transport along the continuous, piecewise smooth loop $(0,0) \to (0,\tau^{\frac{1}{2}}) \to (\tau^{\frac{1}{2}},\tau^{\frac{1}{2}}) \to (\tau^{\frac{1}{2}},0) \to (0,0)$ defines an element $h_\tau \in \mathrm{End}(\mathbb{C}) \cong \mathbb{C}$. Compute h_τ and $\frac{\mathrm{d}h_\tau}{\mathrm{d}\tau}|_{\tau=0}$.

[1] Namely, replace each fiber of E by the set of all orthonormal bases.

[2] Remember that $|\det \mathfrak{m}_{\mathbb{C}}|^2 = \det \mathfrak{m}$