DIFFERENTIAL GEOMETRY I
HOMEWORK 13

DUE: WEDNESDAY, DECEMBER 24

The purpose of this homework set is to study certain covaraint derivative for $T\mathbf{S}^3$ and the corresponding connection on its oriented, orthonormal frame bundle.

- Consider $\mathbf{S}^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2 = 1\}$. Given a smooth function $f : \mathbf{S}^3 \to \mathbb{R}$, extend it to a smooth function on $\mathbb{R}^4\backslash\{0\}$ by $\tilde{f}(\mathbf{x}) = f(\mathbf{x}/|\mathbf{x}|)$. We already know that $\mathrm{d}f = (\mathrm{d}\tilde{f})|_{\mathbf{S}^3}$. With this understood, it is more convenient to calculate the exterior derivative in $\mathbb{R}^4$, and then restrict the output on $\mathbf{S}^3$. Taking the restriction is tantamount to imposing the condition that $x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z + w\,\mathrm{d}w = 0$.
- The tangent bundle of the three dimensional sphere is trivial. This is true only for spheres in some special dimensions. Let
$$\mathbf{e}_1 = (-y,x,-w,z)\ , \qquad \mathbf{e}_2 = (-z,w,x,-y)\ , \qquad \mathbf{e}_3 = (-w,-z,y,x)\ .$$
They constitute an oriented, orthonormal frame for $T\mathbf{S}^3$.
- The dual coframe provides a trivialization for the cotangent bundle. It consists of
$$\begin{aligned}\sigma^1 &= (-y\,\mathrm{d}x + x\,\mathrm{d}y - w\,\mathrm{d}z + z\,\mathrm{d}w)|_{\mathbf{S}^3}\ ,\\ \sigma^2 &= (-z\,\mathrm{d}x + w\,\mathrm{d}y + x\,\mathrm{d}z - y\,\mathrm{d}w)|_{\mathbf{S}^3}\ ,\\ \sigma^3 &= (-w\,\mathrm{d}x - z\,\mathrm{d}y + y\,\mathrm{d}z + x\,\mathrm{d}w)|_{\mathbf{S}^3}\ .\end{aligned}$$
By the same computation as that in the midterm,
$$\mathrm{d}\sigma^1 = 2\,\sigma^2\wedge\sigma^3\ , \qquad \mathrm{d}\sigma^2 = 2\,\sigma^3\wedge\sigma^1\ , \qquad \mathrm{d}\sigma^3 = 2\,\sigma^1\wedge\sigma^2\ .$$
- For any 3×3-matrix $\mathfrak{m}$, let $\tilde{\mathfrak{m}}$ be the following 4×4-matrix
$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & \mathfrak{m} & \\ 0 & & & \end{array}\right] .$$
- It follows that SO(4) is a *trivial* principal SO(3)-bundle over $\mathbf{S}^3$. What follows is an explicit isomorphism:
$$\begin{aligned} F:\quad \mathbf{S}^3\times\mathrm{SO}(3) &\longrightarrow \mathrm{SO}(4) \\ \big((x,y,z,w),g\big) &\longmapsto \begin{bmatrix} x & -y & -z & -w \\ y & x & w & -z \\ z & -w & x & y \\ w & z & -y & x \end{bmatrix}\tilde{g}\ . \end{aligned}$$

- The tangent bundle of $\mathbf{S}^3$ is the associated bundle of SO(4) with the standard representation of SO(3) on $\mathbb{R}^4$. That is to say, $\rho : \mathrm{SO}(3) \to \mathrm{Gl}(3;\mathbb{R})$ is the inclusion map, and $\rho_* : \mathfrak{so}(3) \to \mathbb{M}(3;\mathbb{R})$ is also the inclusion map. The space $\mathfrak{so}(3)$ is the Lie algebra of SO(3), which consists of skew-symmetric 3×3-matrices.
- Any tangent vector field on $\mathbf{S}^3$ can be expressed as $\sum_{j=1}^{3} \psi^j \,\mathbf{e}_j$ where ψ^1, ψ^2, ψ^3 are smooth functions on $\mathbf{S}^3$. We can put them into a column vector, and denote it by ψ. With the identification F, the vector field ψ corresponds to the following SO(3)-equivariant map from $\mathbf{S}^3 \times \mathrm{SO}(3)$ to $\mathbb{R}^3$:

$$((x,y,z,w),g) \overset{\Psi}{\longmapsto} g^{-1}\psi \ .$$

Clearly, Ψ obeys that $\Psi(\,\cdot\,, gh^{-1}) = h\,\Psi(\,\cdot\,, g)$.

(1) For any vector field v, $(\Pi \circ \mathrm{d})(v)$ defines a covariant derivative for $T\mathbf{S}^3$. Here, Π is the orthogonal projection from $\mathbf{S}^3 \times \mathbb{R}^4$ onto $T\mathbf{S}^3$. With respect to the trivialization given by $\{\mathbf{e}_j\}$, the covariant derivative can be expressed as $\mathrm{d} + \mathfrak{a}$ where $\mathfrak{a}$ is a 3×3-matrix with entries being 1-forms. Find out $\mathfrak{a}$.

(2) Calculate the curvature of the covariant derivative in part (1). Your answer shall be a 3×3-matrix whose entries are 2-forms.

(3) Let

$$q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

For $p \in \mathrm{SO}(4)$, $A = q\,p^T\,\mathrm{d}p\,q^T$ defines a $\mathfrak{so}(3)$-valued 1-form on SO(4). Show the covariant derivative for $T\mathbf{S}^3$ induced by A coincides with the one defined in part (1).

(4) Let $\langle\ ,\ \rangle$ be the Riemannian metric induced by the standard inner product of $\mathbb{R}^4$. Denote by ∇ the covariant derivative defined in part (1). Check that

$$\mathrm{d}\langle\psi,\varphi\rangle = \langle\nabla\psi,\varphi\rangle + \langle\psi,\nabla\varphi\rangle \qquad (\star)$$

for any two vector fields ψ and φ.

(5) Bonus Is $(\star)$ true for *any* covariant derivative for $T\mathbf{S}^3$? How about the covariant derivative induced from a $\mathfrak{so}(3)$-connection?

(6) Let M be a smooth manifold. Suppose that v and u are two vector fields on M, and α is a 1-form on M. Prove that

$$(\mathrm{d}\alpha)(v,u) = v(\alpha(u)) - u(\alpha(v)) - \alpha([v,u]) \ .$$

[*Hint*: Compute them in terms of local coordiantes.]