DIFFERENTIAL GEOMETRY I
HOMEWORK 12

DUE: WEDNESDAY, DECEMBER 17

The purpose of this homework set is to study the covariant derivative of some associated bundles of $\mathbf{S}^1 \to \mathrm{SU}(2) \to \mathbf{S}^2$.

- The Lie group G here is $\mathbf{S}^1 \equiv \mathrm{U}(1)$. Its Lie algebra is the space of skew-Hermitian, 1×1-matrices, namely, $i\mathbb{R}$. The exponential map sends it to $e^{it} = \cos t + i \sin t$.
- Regard $\mathbf{S}^1$ as a subgroup of $\mathrm{SU}(2)$ by

$$e^{it} \longmapsto \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} . \tag{0.1}$$

  The convention of the action of $e^{it} \in \mathbf{S}^1$ on $\mathrm{SU}(2)$ is to multiply the *inverse* of the above matrix on the right.
- The Hopf fibration is explained in Homework 11. Identify $(x, y, z) \in \mathbb{R}^3$ with $\begin{bmatrix} iz & -x+iy \\ x+iy & -iz \end{bmatrix}$. The fibration map is given by

$$\begin{aligned} \pi: \quad \mathrm{SU}(2) &\longrightarrow \mathbf{S}^2 \subset \mathbb{R}^3 \\ \mathfrak{m} &\longmapsto \mathfrak{m} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \mathfrak{m}^* . \end{aligned} \tag{0.2}$$

  In other words, it is the image of $(0, 0, 1)$ under the adjoint representation of $\mathrm{SU}(2)$.
- Consider the stereographic projection of $\mathbf{S}^2$:

$$\begin{aligned} (u_1, u_2) \in \mathbb{R}^2 &\overset{\varphi_{\mathcal{U}}}{\longmapsto} \left(\frac{2u_1}{1+|\mathbf{u}|^2}, \frac{2u_2}{1+|\mathbf{u}|^2}, \frac{1-|\mathbf{u}|^2}{1+|\mathbf{u}|^2}\right) \in \mathcal{U} = \mathbf{S}^2 \backslash \{(0,0,-1)\} \\ (v_1, v_2) \in \mathbb{R}^2 &\overset{\varphi_{\mathcal{V}}}{\longmapsto} \left(\frac{2v_1}{1+|\mathbf{v}|^2}, \frac{-2v_2}{1+|\mathbf{v}|^2}, \frac{-1+|\mathbf{v}|^2}{1+|\mathbf{v}|^2}\right) \in \mathcal{V} = \mathbf{S}^2 \backslash \{(0,0,1)\} \end{aligned} \tag{0.3}$$

  where $\mathbf{u} = u_1 + iu_2$ and $\mathbf{v} = v_1 + iv_2$. With the minus sign in front of v_2, the Jacobian matrix has positive determinant. In terms of complex coordinate, the coordinate change is

$$\mathbf{v} = \frac{1}{\mathbf{u}} . \tag{0.4}$$

- To find the bundle transition function $g_{\mathcal{V},\mathcal{U}} : \mathcal{U} \cap \mathcal{V} \to \mathbf{S}^1$, we have to construct local trivializations of $\pi : \mathrm{SU}(2) \to \mathbf{S}^2$ over $\mathcal{U}$ and $\mathcal{V}$. The first step is to construct a section. For instance, we can take

$$\frac{1}{\sqrt{1+|\mathbf{u}|^2}} \begin{bmatrix} i & -\bar{\mathbf{u}} \\ \mathbf{u} & -i \end{bmatrix} \tag{0.5}$$

over $\mathcal{U}$. You can check that its image under π is exactly $\varphi_{\mathcal{U}}(\mathbf{u})$, and hence does define a section over $\mathcal{U}$. Then, we have

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbf{S}^1 & \longrightarrow & \pi^{-1}(\mathcal{U}) \subset \mathrm{SU}(2) \\ (\mathbf{u}, e^{i\alpha}) & \longmapsto & \frac{1}{\sqrt{1+|\mathbf{u}|^2}} \begin{bmatrix} i & -\bar{\mathbf{u}} \\ \mathbf{u} & -i \end{bmatrix} \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} = \frac{1}{\sqrt{1+|\mathbf{u}|^2}} \begin{bmatrix} ie^{i\alpha} & -\bar{\mathbf{u}}e^{-i\alpha} \\ \mathbf{u}e^{i\alpha} & -ie^{-i\alpha} \end{bmatrix} \end{array} . \tag{0.6}$$

Similarly,

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbf{S}^1 & \longrightarrow & \pi^{-1}(\mathcal{V}) \subset \mathrm{SU}(2) \\ (\mathbf{v}, e^{i\beta}) & \longmapsto & \frac{1}{\sqrt{1+|\mathbf{v}|^2}} \begin{bmatrix} i\mathbf{v}e^{i\beta} & -e^{-i\beta} \\ e^{i\beta} & -i\bar{\mathbf{v}}e^{-i\beta} \end{bmatrix} \end{array} . \tag{0.7}$$

It follows that

$$e^{i\beta} = \frac{\sqrt{1+|\mathbf{v}|^2}}{\sqrt{1+|\mathbf{u}|^2}}\mathbf{u}e^{i\alpha} = \frac{\mathbf{u}}{|\mathbf{u}|}e^{i\alpha} . \tag{0.8}$$

In other words,

$$g_{\mathcal{V},\mathcal{U}} = \frac{\mathbf{u}}{|\mathbf{u}|} \tag{0.9}$$

of the above trivializations.

- For any non-negative integer k, let ρ_k be the representation $\rho_k(e^{it}) = e^{-ikt} \in \mathrm{Gl}(1;\mathbb{C})$. Denote by L_k the associated vector bundle $\mathrm{SU}(2) \times_{\rho_k} \mathbb{C}$ over $\mathbf{S}^2$. The bundle transition function for L_k is

$$\rho_k \circ g_{\mathcal{V},\mathcal{U}} = \frac{|\mathbf{u}|^k}{\mathbf{u}^k} \quad (\text{from } \mathcal{U} \text{ to } \mathcal{V}) . \tag{0.10}$$

(1) In terms of local trivializations, sections of L_k are nothing more than $\mathbb{C}$-valued functions on $\mathcal{U}$ and $\mathcal{V}$ that satisfy the bundle transition (0.10). Show that

$$\mathrm{d} + \frac{k}{2}\frac{\mathbf{u}\mathrm{d}\bar{\mathbf{u}} - \bar{\mathbf{u}}\mathrm{d}\mathbf{u}}{1+|\mathbf{u}|^2} \quad \text{over } \mathcal{U} \tag{0.11}$$

can be extended as a covariant derivative of $L_k \to \mathbf{S}^2$. Here, the differential of a complex valued function $f + ig$ is $\mathrm{d}f + i\mathrm{d}g$. Also, write down the covariant derivative over $\mathcal{V}$.

(2) We may write an element $\mathfrak{m} \in \mathrm{SU}(2)$ as $\begin{bmatrix} \mathbf{z} & -\bar{\mathbf{w}} \\ \mathbf{w} & \bar{\mathbf{z}} \end{bmatrix}$ for $(\mathbf{z}, \mathbf{w}) \in \mathbb{C}^2$ and $|\mathbf{z}|^2 + |\mathbf{w}|^2 = 1$. For any integer j with $0 \le j \le k$, check that $\mathfrak{s}_{j,k} : \mathrm{SU}(2) \to \mathbb{C}$ that sends $\mathfrak{m}$ to $\mathbf{z}^j \mathbf{w}^{k-j}$ defines a section of L_k.

(3) Work out the map $\psi : P \times \mathfrak{lie}(G) = \mathrm{SU}(2) \times i\mathbb{R} \to \ker \pi_* \subset TP$. You may write $\mathbf{z} = x_1 + ix_2$ and $\mathbf{w} = x_3 + ix_4$. Then express $\psi(\mathfrak{m}, it)$ as a vector in $\mathbb{R}^4$ which is perpendicular to (x_1, x_2, x_3, x_4).

(4) A connection on P is a linear map A from TP to $\mathfrak{lie}(G)$. In the current case, it is a purely-imaginary valued 1-form on $P = \mathrm{SU}(2)$. Check that

$$\begin{aligned} A &= \frac{1}{2}\,\mathrm{tr}\,\Big(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathfrak{m}^* \mathrm{d}\mathfrak{m} \Big) \\ &= \frac{1}{2}(\bar{\mathbf{z}}\mathrm{d}\mathbf{z} - \mathbf{z}\mathrm{d}\bar{\mathbf{z}} + \bar{\mathbf{w}}\mathrm{d}\mathbf{w} - \mathbf{w}\mathrm{d}\bar{\mathbf{w}})\big|_{\mathbf{S}^3} \\ &= i(-x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2 - x_4\mathrm{d}x_3 + x_3\mathrm{d}x_4)\big|_{\mathbf{S}^3} \end{aligned} \tag{0.12}$$

does define a connection on $\mathrm{SU}(2) \to \mathbf{S}^2$. You have to check those two conditions in [T; §11.4.4]. For the first condition, it is a straightforward computation by using part (3). For the second condition, note that G is abelian here.

(5) A section of L_k can be expressed as two smooth functions $s_1(\mathbf{u})$ and $s_2(\mathbf{v})$ that satisfy the bundle transition (0.10). Consider

$$\begin{aligned} \mathfrak{s}_1 :\ & \mathbb{R}^2 \times \mathbf{S}^1 \longrightarrow \mathbb{C} \\ & (\mathbf{u}, e^{i\alpha}) \longmapsto e^{ik\alpha} s_1(\mathbf{u}) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathfrak{s}_2 :\ & \mathbb{R}^2 \times \mathbf{S}^1 \longrightarrow \mathbb{C} \\ & (\mathbf{v}, e^{i\beta}) \longmapsto e^{ik\beta} s_2(\mathbf{v}) \end{aligned}$$

With (0.6), (0.7) and (0.9), $\mathfrak{s}_1$ and $\mathfrak{s}_2$ together define a smooth function on $\mathrm{SU}(2)$, and correspond to the section of L_k given by s_1 and s_2. Work out the covariant derivative of s_1 by using $\mathfrak{s}_1$ and A (see (0.12)), and show that it coincides with (0.11). It suffices to figure out $\nabla_{\frac{\partial}{\partial u_1}} s_1$ and $\nabla_{\frac{\partial}{\partial u_2}} s_1$.