DIFFERENTIAL GEOMETRY I
HOMEWORK 11

DUE: WEDNESDAY, DECEMBER 10

(1) Let G be a *compact*, *connected* Lie group. Let ω be a *lef-invaraint* volume form of G. It is unique up to a constant multiple.

(a) Show that ω is also right-invariant, and thus bi-invariant. [*Hint*: For any $g \in G$, it is clear that $r_g^*\omega$ is still a left-invariant volume form. It follows that $r_g^*\omega$ is a constant multiple of ω, and denote this constant by $f(g)$. That is to say, $f(g) = \frac{r_g^*\omega}{\omega}$ is a smooth function from G to $\mathbb{R}\backslash\{0\}$. Verify that $f(gh) = f(g)f(h)$. It follows that $f(G)$ is a compact, connected subgroup of $(\mathbb{R}\backslash\{0\}, \times)$.]

(b) Let $\langle \, , \, \rangle$ be a left invariant metric[1] on G. Define a new Riemannian metric on G by

$$\langle\!\langle u, v \rangle\!\rangle_h = \int_G \langle (r_g)_*(u), (r_g)_*(v) \rangle_{hg}\, \omega$$

for any $u, v \in T_hG$. Roughly speaking, $\langle\!\langle \, , \, \rangle\!\rangle$ at h is obtained by averaging $\langle \, , \, \rangle$ over $\{hg \mid g \in G\}$. Show that $\langle\!\langle \, , \, \rangle\!\rangle$ is a bi-invariant metric. [*Hint*: Let M be an oriented n-dimensional manifold, and σ be a n-form on M. Suppose that $f : M \to M$ is a orientation preserving diffeomoprhism. It follows from the change of variable formula that $\int_M f^*\sigma = \int_M \sigma$. This fact can also be used to prove part (a).]

As a result, any compact Lie group admits a bi-invariant metric.

(2) This exercise is the correct version of [T; Example 10.4]. Consider the Lie group

$$\mathrm{SU}(2) = \left\{\mathfrak{m} \in \mathrm{Gl}(2;\mathbb{C}) \;\middle|\; \mathfrak{m}^*\mathfrak{m} = \mathbf{I} \text{ and } \det \mathfrak{m} = 1\right\} ,$$

and identify the space $\mathbb{R}^3$ as its Lie algebra by

$$(x, y, z) \in \mathbb{R}^3 \;\longleftrightarrow\; \mathfrak{X} = \begin{bmatrix} iz & -x + iy \\ x + iy & -iz \end{bmatrix} .$$

(a) Consider the representation of $\rho : \mathrm{SU}(2) \to \mathrm{Gl}(3;\mathbb{R})$ defined by

$$\rho(\mathfrak{m})\mathfrak{X} = \mathfrak{m}\mathfrak{X}\mathfrak{m}^* .$$

where right hand side is the product of three 2×2-matrices. Check that the image of ρ actually belongs to $\mathrm{SO}(3)$.

(b) Show that $\rho(\mathrm{SU}(2))$ acts on the unit sphere $\mathbf{S}^2 \subset \mathbb{R}^3$ transitively.

(c) Find out the stabilizer of $(0, 0, 1) \in \mathbf{S}^2$.

[1] To avoid the confusion with the elements of the Lie group, we do not use $g(\, , \,)$ for the notation of the Riemannian metric here.

It follows that SU(2) constitute a principal $\mathbf{S}^1$-bundle over $\mathbf{S}^2$, $\mathbf{S}^1 \to \mathrm{SU}(2) \cong \mathbf{S}^3 \to \mathbf{S}^2$. This is called the *Hopf fibration.*

(3) This is a continuation of Exercise (2). For any $k \in \mathbb{Z}$, let $\rho_k : \mathbf{S}^1 \to \mathrm{SO}(2) \subset \mathrm{Gl}(2;\mathbb{R})$ be the representation defined by

$$\rho_k(e^{i\theta}) = \begin{bmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{bmatrix} .$$

Here, we regard $\mathbf{S}^1$ as $\mathbb{R}^1/2\pi\mathbb{Z}$. Let E_k be associated vector bundle of $\mathrm{SU}(2) \to \mathbf{S}^2$ with ρ_k, namely, $E_k = \mathrm{SU}(2) \times_{\rho_k} \mathbb{R}^2$. It is a rank 2 vector bundle over $\mathbf{S}^2$.

(a) Show that E_k is isomorphic to E_{-k}. [*Remark*: They are not the same as *oriented* vector bundles. To be more precise, there is a natural way to assign a bundle orientation to them. And there is no orientation-preserving bundle isomorphism between E_k and E_{-k} for $k \neq 0$.]

(b) Prove that E_2 is isomorphic to $T\mathbf{S}^2$. [*Hint*: Work out the bundle transition functions of them. For $T\mathbf{S}^2$, you may take the round metric, and find out the bundle transition as a smooth map from $U \cap V$ to SO(2).]