DIFFERENTIAL GEOMETRY I
HOMEWORK 10

DUE: WEDNESDAY, DECEMBER 3

Lie Derivative of Vector Fields

Commutator of derivations. Suppose that U and V are two vector fields on M. Consider their *Lie bracket*

$$[U,V] := UV - VU \ . \tag{a}$$

The equation is understood as an operator on $\mathcal{C}^\infty(M;\mathbb{R})$. Remember that a vector field U acts on a smooth function f by $U(f) = (\mathrm{d}f)(U)$.

(i) Check that $[U,V]$ is still a derivation. It is clear that the operator is linear over $\mathbb{R}$, and it suffices to check the Leibniz property.

(ii) It follows that $[U,V]$ is still a vector field. On a coordinate chart, U and V can be expressed as $\sum_i u^i(x)\frac{\partial}{\partial x^i}$ and $\sum_i v^i(x)\frac{\partial}{\partial x^i}$, respectively. Work out the expression of $[U,V]$ on the coordinate chart.

(iii) It follows from the definition that $[U,V] = -[V,U]$. Check that the Lie bracket obeys the *Jacobi identity*, namely,

$$[U,[V,W]] + [W,[U,V]] + [V,[W,U]] = 0 \ . \tag{b}$$

[*Hint*: It is a direct computation in terms of a local coordinate.]

(iv) (You shall check the following property yourself, but you <u>don't</u> have to submit this part.) Suppose that there is a *diffeomorphism* $\psi : M \to N$. In general, the push-forward map ψ_* sends a vector field on M to a section of ψ^*TN (see [T; §5.3]). When ψ is a diffeomorphism, we can think that ψ_* sends a vector field on M to a vector field on N. With this understood,

$$\psi_*([U,V]) = [\psi_*U, \psi_*V] \ . \tag{c}$$

[*Hint*: In terms of a local coordinate, it is nothing more than the chain rule.]

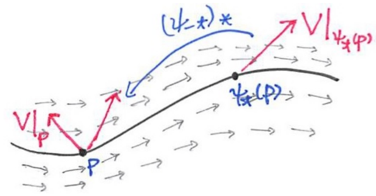
Lie derivative of vector fields. Given a vector field U on M, it associates a one-parameter family of self-diffeomorphisms ψ_t of M defined by

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}t} = U \quad \text{(or more precisely, } \frac{\mathrm{d}\psi_t(p)}{\mathrm{d}t} = U|_{\psi_t(p)} \ \text{ for any } p \in M)$$

with ψ_0 to be the identity map. On a coordinate chart, $\psi_t = (\psi_t^1(x), \ldots, \psi_t^n(x))$ is the solution of

$$\frac{\mathrm{d}\psi_t^j(x)}{\mathrm{d}t} = u^j(\psi_t(x)) \ \text{ with } \psi_0(x) = x \ . \tag{d}$$

By the fundamental theorem of O.D.E., there exists $\epsilon > 0$ such that the solution exists for $t \in (-\epsilon, \epsilon)$, and is smooth in x. It follows from the uniqueness of the solution for an O.D.E. that $\psi_{t_1} \circ \psi_{t_2} = \psi_{t_1+t_2}$. Therefore, when M is compact, $\psi_t(x)$ can be defined for any $t \in \mathbb{R}$. And the inverse map of ψ_t is ψ_{-t}. It justifies the name of *one-parameter family* of *self-diffeomorphisms.*



(v) Find out the one-parameter family of diffeomorphisms generated by $-y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$ on $\mathbb{R}^2$.

(vi) Let ψ_t be the one-parameter family of diffeomorphisms generated by a vector field U. The *Lie derivative* of V with respect to U is defined as follows

$$(L_U V)|_p = \lim_{t\to 0} \frac{(\psi_{-t})_*(V|_{\psi_t(p)}) - V|_p}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big((\psi_{-t})_*(V|_{\psi_t(p)})\big) \tag{e}$$

where the first term in the enumerator means the push-forward of $V|_{\psi_t(p)}$ by ψ_{-t}. Since $\psi_{-t}(\psi_t(p)) = p$, $(\psi_{-t})_*(V|_{\psi_t(p)})$ is a tangent vector at p, and $(L_U V)|_p$ is the derivative of a map from $(-\epsilon, \epsilon)$ to the vector space T_pM. Show that $L_U V = [U, V]$. [*Hint*: In terms of a local coordinate, $(\psi_{-t})_*(V|_{\psi_t(p)})$ is $v^i(\psi_t(p))\dfrac{\partial \psi^j_{-t}}{\partial x^i}\Big|_p \dfrac{\partial}{\partial x^j}$. The Lie derivative (e) can be found by differentiating the coefficient functions with respect to t and evaluating at $t = 0$. You shall use the defining equation (d) of ψ_t.]

Revisiting the Jacobi identity.

(vii) Let ψ_t be the one-parameter family of diffeomorphisms generated by a vector field U. Apply part (iv) and (vi) to give another proof for the Jacobi identity (b). [*Hint*: Differentiate $(\psi_{-t})_*([V, W]) = [(\psi_{-t})_*V, (\psi_{-t})_*W]$ with respect to t.]

Lie group and Lie algebra

Let G be a Lie group, and $\mathfrak{g}$ be its tangent space at the identity. There is a one-to-one correspondence between vectors in $\mathfrak{g}$ and left-invariant vector fields on G. Due to (c), the Lie bracket between two left-invariant vector fields is still left-invariant. Therefore, the Lie bracket induces a binary operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Definition. A *Lie algebra* (over $\mathbb{R}$) is a vector space together with a binary operation which is bilinear and satisfies the Jacobi identity (b). The tangent space at the identity of a Lie group is an example of a Lie algebra.

Example. The space $\mathbb{R}^3$ with the standard cross product constitutes a Lie algebra. In fact, it is isomorphic to the Lie algebra of SO(3).

(viii) Consider $G = \mathrm{Gl}(n;\mathbb{R})$ and $\mathfrak{g} = \mathbb{M}(n;\mathbb{R})$. For any $\mathfrak{a}, \mathfrak{b} \in \mathbb{M}(n;\mathbb{R})$, check that their Lie bracket coincides with the usual matrix bracket. [*Hint*: Let $\mathfrak{m}_{ij}$ be the standard coordinate for the space of $n \times n$-matrices. The vector $\mathfrak{a}$ corresponds to the left-invariant vector field $\mathrm{tr}(\mathfrak{a}^T\mathfrak{m}^T\frac{\partial}{\partial\mathfrak{m}}) = \mathfrak{a}_{ji}\mathfrak{m}_{kj}\frac{\partial}{\partial\mathfrak{m}_{ki}}$. Note that at the identity $\mathfrak{m}_{kj} = \delta_{kj}$, $\mathrm{tr}(\mathfrak{a}^T\mathfrak{m}^T\frac{\partial}{\partial\mathfrak{m}}) = \mathfrak{a}_{ji}\frac{\partial}{\partial\mathfrak{m}_{ji}}$. The bracket can be computed directly from the coordinate expression.]

Remark. If one uses *right-invariant* vector fields to construct the Lie bracket on $\mathfrak{g}$, it differs from the left-invariant one by a minus sign. These two Lie algebras are isomorphic to each other. The isomorphism is given by the differential of the inverse map, $g \mapsto g^{-1}$, at the identity. Since the bracket of the left-invariant construction coincides with the matrix bracket, the usual convention is the left-invariant one.

Exponential map on a Lie group. On a Lie group G, the exponential map $\exp : \mathfrak{g} \to G$ is defined as follows. For any $\mathfrak{a} \in \mathfrak{g}$, let A be the left-invariant vector field corresponding to $\mathfrak{a}$. That is to say, $A|_g = (\ell_g)_*\mathfrak{a} \in T_gG$. Let ψ_t be the one-parameter family of diffeomorphisms generated by the vector field A. The exponential of $\mathfrak{a}$ is defined to be $\psi_1(\mathbf{I})$, and is usually denoted by $e^{\mathfrak{a}}$. Namely, flow the identity along ψ_t for time 1.

It follows that $e^{t\mathfrak{a}}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{t\mathfrak{a}} = (\ell_{e^{t\mathfrak{a}}})_*\mathfrak{a} \ \in T_{e^{t\mathfrak{a}}}G \ . \tag{f}$$

By the uniqueness of the solution for an O.D.E., this equation characterizes $e^{t\mathfrak{a}}$. Recall that for matrix groups, we checked that the matrix exponential map obeys (f). Hence, the two definitions for the exponential map coincide for matrix groups.

(ix) Find the expression of $\psi_t(g)$ in terms of g and $e^{t\mathfrak{a}}$. [*Hint*: Intuitively, it should be either $ge^{t\mathfrak{a}}$ or $e^{t\mathfrak{a}}g$.]