DIFFERENTIAL GEOMETRY I
HOMEWORK 9

DUE: WEDNESDAY, NOVEMBER 19

(1) Consider the n-sphere $\mathbf{S}^n = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{j=1}^{n+1}(x^j)^2 = 1\}$ with the metric g to be the restriction of $\sum_{j=1}^{n+1}(\mathrm{d}x^j)^2$. Let $p = (0, \ldots, 0, 1)$, then $T_p\mathbf{S}^2$ can be identified with the hyperplane $\{(u^1, \ldots, u^{n+1}) \in \mathbb{R}^{n+1} \mid u^{n+1} = 0\}$.

(a) For $\mathbf{u} = (u^1, \ldots, u^n) \in T_p\mathbf{S}^n$ with $|\mathbf{u}|^2 = \sum_{j=1}^{n}(u^j)^2 < \pi^2$, find out $\exp_p \mathbf{u}$.

(b) Calculate the metric g_{ij} of the Gaussian coordinate centered at p, and check that

$$\sum_{j=1}^{n}(g_{ij}(\mathbf{u}) - \delta_{ij})u^j = 0 \quad \text{and} \quad |g_{ij}(\mathbf{u}) - \delta_{ij}| \leq c|\mathbf{u}|^2 \ .$$

(2) Consider the disk $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ with the Poincaré metric

$$g = \frac{4}{(1 - x^2 - y^2)^2}\big((\mathrm{d}x)^2 + (\mathrm{d}y)^2\big) \ .$$

Let p be the origin, then T_pB is naturally identified with the xy-plane.

(a) For $\mathbf{v} = (v^1, v^2) \in \mathbb{R}^2$, identify it with $\frac{1}{2}(v^1\frac{\partial}{\partial x} + v^2\frac{\partial}{\partial y}) \in T_pB$. Find out $\exp_p \mathbf{v}$. [*Hint*: In Homework 8, you almost find the geodesics. Also, note that $g|_p = (2\mathrm{d}x)^2 + (2\mathrm{d}y)^2$.]

(b) Calculate the metric g_{ij} of the Gaussian coordinate centered at p, and check that

$$\sum_{j=1}^{2}(g_{ij}(\mathbf{v}) - \delta_{ij})v^j = 0 \quad \text{and} \quad |g_{ij}(\mathbf{v}) - \delta_{ij}| \leq c|\mathbf{v}|^2$$

on a small neighborhood of the origin.