DIFFERENTIAL GEOMETRY I
HOMEWORK 8

DUE: WEDNESDAY, NOVEMBER 12

Models for Hyperbolic Geometry

Hyperboloid model. Let $\mathbb{R}^3$ be equipped with the following non-degenerate, symmetric bilinear form:

$$\langle\!\langle x, y \rangle\!\rangle = x_1 y_1 + x_2 y_2 - x_3 y_3 \ . \tag{†}$$

Another gadget we are going to use is the following symmetric $(0,2)$-tensor on $\mathbb{R}^3$:

$$(\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 - (\mathrm{d}x_3)^2 \in \mathcal{C}^\infty(\mathbb{R}^3; \mathrm{Sym}^2 T^*\mathbb{R}^3) \ . \tag{‡}$$

Consider one-branch of the two-sheeted hyperboloid:

$$H = \big\{ x \in \mathbb{R}^3 \ \big| \ \langle\!\langle x, x \rangle\!\rangle = -1 \text{ and } x_3 > 0 \big\} \ .$$

Since the only critical (non-regular) value of $\langle\!\langle x, x \rangle\!\rangle$ is 0. The set H is a submanifold of $\mathbb{R}^3$, and $T_x H = \{ v \in \mathbb{R}^3 \mid \langle\!\langle x, v \rangle\!\rangle = 0 \}$.

It is a straightforward computation to check that

$$\begin{array}{cccc} \psi : & \mathbb{R}^2 & \to & \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^2 \\ & u & \mapsto & (u, \sqrt{1 + u^2}) \end{array}$$

defines a diffeomorphism from $\mathbb{R}^2$ to H.

(i) Let g be the restriction of (‡) on H. Show that g is a Riemannian metric on H by checking that $\psi^* g$ is positive-definite.

(ii) Calculate the Christoffel symbols for $(\mathbb{R}^2, \psi^* g)$. Then, write down its geodesic equation, and check that $u(t) = \sinh(t)\underline{v}$ is a geodesic for any $\underline{v} \in \mathbb{R}^2$ of unit length. Note that the initial point $u(0) = 0$, and the initial velocity $u'(0) = \underline{v}$.

It follows that $\psi(u(t)) = (\sinh(t)\underline{v}, \cosh(t))$ is a geodesic on (H, g).

Isometry and geodesic for the hyperboloid model. Introduce the *Lorentz group*:

$$\mathrm{O}(2,1) = \big\{ \mathfrak{m} \in \mathrm{Gl}(3; \mathbb{R}) \ \big| \ \mathfrak{m}^T \, \mathfrak{L} \, \mathfrak{m} = \mathfrak{L} \big\}$$

where

$$\mathfrak{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \ .$$

(iii) For any $\mathfrak{m} \in \mathrm{O}(2,1)$, regard it as a self-diffeomorphism of $\mathbb{R}^3$. Check that $\langle\!\langle \ , \ \rangle\!\rangle$ is $\mathfrak{m}$-invariant, and the pull-back of (‡) under $\mathfrak{m}$ is still (‡). [*Hint*: Indeed, these two statements are the same.]

It has the following consequence. Any $\mathfrak{m} \in \mathrm{O}(2,1)$ induces a self-diffeomorphism of the two-sheeted hyperboloid $\{x \in \mathbb{R}^3 \mid \langle\langle x, x \rangle\rangle = -1\}$. Thus, $\mathfrak{m}(H)$ is either H itself, or another branch of the hyperboloid. If $\mathfrak{m}$ preserves H, it is an *isometry*[1] of (H, g). Clearly, an isometry maps a geodesic onto another geodesic.

(iv) For any $x \in H$ and $v \in T_xH$, show that the geodesic with initial position x and unit-normed initial velocity v is $\cosh(t)x + \sinh(t)v$. [*Hint*: We already finished the case when $x = (0,0,1)$. Assume that $x \neq (0,0,1)$. Write x as $(\sinh\rho\cos\theta, \sinh\rho\sin\theta, \cosh\rho)$ with $\rho > 0$. Consider the matrix $\mathfrak{m}$ with the first column to be $(\cosh\rho\cos\theta, \cosh\rho\sin\theta, \sinh\rho)$, the second column to be $(-\sin\theta, \cos\theta, 0)$, and the third column to be x. Those two vectors are obtained by normalizing $\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial\theta}$ with respect to (‡). The linear map $\mathfrak{m}$ sends $(0,0,1)$ to x.]

(v) Conclude that the geodesics of (H, g) are exactly given by the intersection of H with a two-plane (two-dimensional vector subspace of $\mathbb{R}^3$).

Poincaré model. Let φ be the *pseudo-inversion* with pole $q = (0,0,-1)$ of modulus -2:

$$\varphi(x) = q - \frac{2(x-q)}{\langle\langle x-q, x-q \rangle\rangle} .$$

It obeys $\langle\langle \varphi(x) - q, x - q \rangle\rangle = -2$, which is analogous to the classical inversion relation.

(vi) Show that φ is a diffeomorphism from H onto the unit disk $B = \{s \in \mathbb{R}^2 \mid |s| < 1\}$ of the xy-plane, and write down the explicit formula for $(\varphi^{-1})^*g$.

(vii) Describe *geometrically* the geodesics of $(B, (\varphi^{-1})^*g)$. [*Hint*: It is easier to use Part (v).]

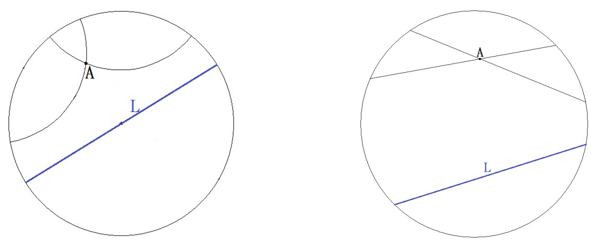
Beltrami–Klein model. There is another diffeomorphism from H to B. Consider the affine plane $P = \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$. For any $x \in H$, $\pi(x)$ is the intersection of P with the line passing through the origin and x.

(viii) Show that π is a diffeomorphism from H onto the unit disk B, and write down the explicit formula for $(\pi^{-1})^*g$.

(ix) Describe *geometrically* the geodesics of $(B, (\pi^{-1})^*g)$. [*Hint*: It is easier to use Part (v).]

Isometry group. All the isometries of a Riemannian manifold constitute a group, where the binary operation is the composition. This group is called the *isometry group*.

(x) $\boxed{\text{bonus}}$ Explain the isometry group for the Poincaré model and the Beltrami–Klein model.



The hyperbolic geometry does not obey the parallel postulate (the fifth postulate in Euclid's Elements). The study of non-Euclidean geometry is motivated by understanding the parallel postulate. It eventually leads to the creation of Riemannian geometry.

[1] An isometry of a Riemannian manifold is a self-diffeomorphism which pulls back the metric to itself.