DIFFERENTIAL GEOMETRY I
HOMEWORK 7

DUE: WEDNESDAY, NOVEMBER 5

(1) The *Heisenberg group* is diffeomorphic to $\mathbb{R}^3$, but the Lie group structure is different. It is not abelian.

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \; : \; x, y, z \in \mathbb{R} \right\} .$$

It is not hard to show that H is a Lie subgroup of $\mathrm{Gl}(3;\mathbb{R})$. Denote its tangent space at the identity by $\mathfrak{h}$. The space $\mathfrak{h}$ can be identified with the space of 3×3 matrices whose lower-diagonal elements all vanish.

(a) Calculate $\exp : \mathfrak{h} \to H$, and show that it is a diffeomorphism. [*Hint*: You may use u, v, w as the coordinates for $\mathfrak{h}$ to avoid confusion. The power series in this case turns out to have only finitely many terms. The inverse map of exponential can be solved explicitly.]

(b) Construct three linearly independent *left-invariant* 1-forms on H. [*Hint*: Any 1-form on H can be expressed as $a_1\mathrm{d}x + a_2\mathrm{d}y + a_3\mathrm{d}z$ for $a_j \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R})$.]

(c) Construct three linearly independent *right-invariant* 1-forms on H.

(d) Use Part (b) and (c) to construct a *bi-invariant*[1] volume form on H.

Remark. With Part (b), it is fairly easy to construct a left-invariant Riemannian metric on H. Let σ^1, σ^2 and σ^3 be the answer to Part (b). Choose a positive-definite, symmetric 3×3 matrix $[b_{ij}]_{i,j=1}^3$. Then, $\sum_{i,j=1}^3 b_{ij}\, \sigma^i \otimes \sigma^j$ defines a left-invariant Riemannian metric on H. Such a construction works for any Lie group.

(2) Consider

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \; : \; x > 0 \text{ and } y \in \mathbb{R} \right\} .$$

Its tangent space at the identity can be identified with

$$\mathfrak{g} = \left\{ \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} \; : \; u, v \in \mathbb{R} \right\} .$$

A direct computation shows that

$$\exp(\begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} e^u & \dfrac{e^u - 1}{u} v \\ 0 & 1 \end{bmatrix} .$$

[1] A differential form on a Lie group is called bi-invariant if it is both left-invariant and right-invariant.

(a) Construct two linearly independent left-invariant 1-forms on G, and calculate their pull-back under the exponential map. [*Hint*: The function $1/x$ is smooth on G.]

(b) Construct a left-invariant and a right-invariant volume form on G, and show that the only bi-invariant 2-form on G is zero.

Remark. In general, a Lie group may not admit a bi-invariant volume form.

(3) Consider $\mathrm{Sl}(2;\mathbb{R}) = \{\mathfrak{m} \in \mathrm{Gl}(2;\mathbb{R}) \mid \det \mathfrak{m} = 1\}$. Denote the identity matrix by $\mathbf{1}$. Its tangent space at $\mathbf{1}$ can be identified with 2×2, traceless matrices. Denote it by

$$\mathfrak{sl}(2;\mathbb{R}) = \big\{\mathfrak{a} \in \mathbb{M}(2;\mathbb{R}) \mid \mathrm{tr}(\mathfrak{a}) = 0\big\} .$$

Consider the exponential map

$$\begin{array}{rccl} \exp: & \mathfrak{sl}(2;\mathbb{R}) & \to & \mathrm{Sl}(2;\mathbb{R}) \\ & \mathfrak{a} & \mapsto & \mathbf{1} + \mathfrak{a} + \frac{1}{2}\mathfrak{a}^2 + \cdots + \frac{1}{k!}\mathfrak{a}^k + \cdots \end{array} .$$

By using #1 of Homework 2, it is not hard to show that the image does belong to $\mathrm{Sl}(2;\mathbb{R})$.

(a) Prove that for any $\mathfrak{a} \in \mathfrak{sl}(2;\mathbb{R})$, the eigenvalues of $\exp(\mathfrak{a})$ lie either in the unit circle, or in the positive real line. [*Hint*: Consider the Jordan form of $\mathfrak{a}$.]

It follows that $-\mathbf{1} \in \mathrm{Sl}(2;\mathbb{R})$ does not belong to the image of the exponential map, and the exponential map is *not surjective*.

(b) For any $\mathfrak{a} \in \mathfrak{sl}(2;\mathbb{R})$, prove that

$$\exp(\mathfrak{a}) = (\cosh\lambda)\,\mathbf{1} + \frac{\sinh\lambda}{\lambda}\mathfrak{a}$$

where $\lambda = (-\det\mathfrak{a})^{\frac{1}{2}}$ is one of the eigenvalues of $\mathfrak{a}$. When $\det\mathfrak{a} = 0$, the above formula reads $\exp(\mathfrak{a}) = \mathbf{1} + \mathfrak{a}$. [*Hint*: Consider the Jordan form of $\mathfrak{a}$. Remember that the hyperbolic sine and cosine functions are nothing more than simple linear combinations of the exponential functions.]

For $\mathfrak{a} = \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} \in \mathfrak{sl}(2;\mathbb{R})$, $\exp(\mathfrak{a}) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Thus, the exponential map is *not injective*. The inverse function theorem says that there exists a neighborhood U of $\mathbf{0}$ such that the restriction of the exponential map on U is a diffeomorphism onto its image. In this case, we may take that U to be $\{\mathfrak{a} \in \mathfrak{sl}(2;\mathbb{R}) \mid (\det\mathfrak{a})^2 < \pi^2\}$.

(4) Reading assignment: [T; §5.7].