DIFFERENTIAL GEOMETRY I
HOMEWORK 6

DUE: WEDNESDAY, OCTOBER 29

(1) Consider the following $(n-1)$-form on $\mathbb{R}^n \backslash \{\mathbf{0}\}$:

$$\omega = \sum_{j=1}^{n} \frac{(-1)^{j+1} x^j \, \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n}{\big((x^1)^2 + \cdots + (x^n)^2\big)^{\frac{n}{2}}}$$

where $\widehat{\mathrm{d}x^j}$ means that $\mathrm{d}x^j$-term is not there.

(a) Check that ω is closed.

(b) Prove that ω is not exact. [*Hint*: Consider the integral of ω on $\mathbf{S}^{n-1}$. Note that $\omega|_{\mathbf{S}^{n-1}} = (|\mathbf{x}|^n \omega)|_{\mathbf{S}^{n-1}}$. You may invoke the Stokes theorem on the integration of the latter $(n-1)$-form over $\mathbf{S}^{n-1}$.]

(2) The main purpose of this exercise is to prove that $\mathrm{H}^1_{\mathrm{d}}(\mathbf{S}^2)$ is trivial. Namely, any closed 1-form on $\mathbf{S}^2$ must be exact.

(a) Show that any closed 1-form on $\mathbb{R}^2$ is exact. [*Hint*: Let σ be a closed 1-form on $\mathbb{R}^2$. For any $P \in \mathbb{R}^2$, integrate σ along a rectangular (directed) path from $\mathbf{0}$ to P. Does it depend on the choice of the path?]

(b) Show that any closed 1-form on $\mathbf{S}^2$ is exact.

(3) Consider the two-torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$. The main purpose of this exercise is to show that $\mathrm{H}^1_{\mathrm{d}}(\mathbf{T}^2) \cong \mathbb{R}^2$. The following map is the quotient map from $\mathbb{R}^2$ to $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$:

$$\begin{array}{cccc} \pi: & \mathbb{R}^2 & \to & \mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1 \\ & (\alpha, \beta) & \mapsto & (e^{2\pi i \alpha}, e^{2\pi i \beta}) \end{array} .$$

There exist two trivializing sections for the cotangent bundle of $\mathbf{T}^2$, whose pull-back under π are $\mathrm{d}\alpha$ and $\mathrm{d}\beta$, respectively. They are usually denoted by $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ on $\mathbf{T}^2$. It is an abuse of notation, but turns out to be quite convenient. Any 1-form on $\mathbf{T}^2$ can be written as $g\,\mathrm{d}\alpha + h\,\mathrm{d}\beta$ for $g, h \in \mathcal{C}^\infty(\mathbf{T}^2; \mathbb{R})$. Note that the notation does *not* suggest that $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ are exact on $\mathbf{T}^2$.

(a) For any $\beta \in \mathbb{R}$, consider

$$\begin{array}{cccc} \gamma_\beta: & \mathbf{S}^1 & \to & \mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1 \\ & e^{2\pi i \alpha} & \mapsto & (e^{2\pi \alpha}, e^{2\pi i \beta}) \end{array} .$$

Let σ be a closed 1-form on $\mathbf{T}^2$. Prove that $\int_{\mathbf{S}^1} \gamma_\beta^* \sigma$ is independent of β, where $\mathbf{S}^1$ is oriented counterclockwisely.

(b) Let σ be a closed 1-form on $\mathbf{T}^2$. Prove that $\sigma - (\int_{\mathbf{S}^1} \gamma_\beta^*\sigma)\mathrm{d}\alpha - (\int_{\mathbf{S}^1} \gamma_\alpha^*\sigma)\mathrm{d}\beta$ is exact, where γ_α is defined similarly. [*Hint*: In any event, $\pi^*\sigma$ is a closed 1-form on $\mathbb{R}^2$, and is exact due to Part (a) of #2. However, σ is exact only when that function on $\mathbb{R}^2$ can taken to be 1-periodic in both α and β variables.]

The above argument shows that $\dim \mathrm{H}^1_{\mathrm{d}}(\mathbf{T}^2) \leq 2$. A similar argument as that in #2 of Homework 4 shows that any linear combination of $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ (with rational coefficients) cannot be exact. It follows that $\mathrm{H}^1_{\mathrm{d}}(\mathbf{T}^2) \cong \mathbb{R}^2$.

Since the de Rham cohomology is invariant under diffeomorphism, it shows that $\mathbf{S}^2$ is not diffeomorphism to $\mathbf{T}^2$.

(4) Construct three 2×2 matrices with real entries, A, B and C such that $\mathrm{tr}(ABC) \neq \mathrm{tr}(BAC)$.

(5) [Suggested reading, not a writing homework] Let M be a *compact* manifold for simplicity, and $\pi : E \to M$ be a rank k vector bundle. Then

- E can be realized as a subbundle of the trivial bundle $M \times \mathbb{R}^\ell$ for any sufficiently large ℓ;
- for any sufficiently large ℓ, there exists a map $\psi : M \to \mathbf{Gr}(\ell; k)$ such that the pull-back of the tautological bundle is E.

The first item is in the last paragraph of [T; §4.1], and the second item is in [T; §5.2]. The second item means that the information of E can be encoded in a map from M to a Grassmannian manifold.

(6) [Appendix] Let $\mathfrak{a}$ and $\mathfrak{b}$ be two $n \times n$ matrices. What follows is another argument for

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{\mathfrak{a}+t\mathfrak{b}} = \int_0^1 e^{(1-s)\mathfrak{a}}\, \mathfrak{b}\, e^{s\mathfrak{a}}\, \mathrm{d}s\ . \tag{†}$$

To start, note that

$$\frac{\mathrm{d}}{\mathrm{d}s} e^{s\mathfrak{c}} = e^{s\mathfrak{c}}\, \mathfrak{c} = \mathfrak{c}\, e^{s\mathfrak{c}}$$

for any $\mathfrak{c} \in \mathbb{M}(n;\mathbb{R})$. Now,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(e^{-\mathfrak{a}} e^{\mathfrak{a}+t\mathfrak{b}}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{-s\mathfrak{a}} e^{s(\mathfrak{a}+t\mathfrak{b})}\Big)\,\mathrm{d}s \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\big(e^{-s\mathfrak{a}} e^{s(\mathfrak{a}+t\mathfrak{b})}\big)\,\mathrm{d}s \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(-e^{-s\mathfrak{a}}\,\mathfrak{a}\, e^{s(\mathfrak{a}+t\mathfrak{b})} + e^{-s\mathfrak{a}}\,(\mathfrak{a}+t\mathfrak{b}) e^{s(\mathfrak{a}+t\mathfrak{b})}\big)\,\mathrm{d}s \\
&= \int_0^1 \Big(-e^{-s\mathfrak{a}}\,\mathfrak{a}\,\big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{s(\mathfrak{a}+t\mathfrak{b})}\big) \\
&\qquad + e^{-s\mathfrak{a}}\,\mathfrak{b}\, e^{s\mathfrak{a}} + e^{-s\mathfrak{a}}\,\mathfrak{a}\,\big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{s(\mathfrak{a}+t\mathfrak{b})}\big)\Big)\,\mathrm{d}s
\end{aligned}$$

which finishes the proof for (†).