DIFFERENTIAL GEOMETRY I
HOMEWORK 5

DUE: WEDNESDAY, OCTOBER 22

(1) Consider $\mathbf{S}^2$ with the stereographic projection (see #4 of Homework 4). For any $\alpha \in \mathbb{R}$,

$$s_\alpha = \big(1 + (u^1)^2 + (u^2)^2\big)^\alpha \mathrm{d}u^1 \wedge \mathrm{d}u^2$$

defines a section of $\Lambda^2 T^*\mathbf{S}^2$ over U. For which value of α can s_α be extended as a nonwhere vanishing global section?

Once you find such α, it gives you a nowhere vanishing 2-form on $\mathbf{S}^2$. Therefore, $\Lambda^2 T^*\mathbf{S}^2$ is isomorphic to the trivial bundle $\mathbf{S}^2 \times \mathbb{R}$.

(2) Consider the tautological bundle over $\mathbb{RP}^2$:

$$E = \big\{([\mathbf{x}], \mathbf{v}) \in \mathbb{RP}^2 \times \mathbb{R}^3 \;\big|\; |\mathbf{x}| = 1, \mathbf{v} \text{ is parallel to } \mathbf{x}\big\}$$

with the following local trivializations

$$\begin{aligned}
\psi_1 &: ((x, y), \alpha) \mapsto \big([x, y, 1], (x\alpha, y\alpha, \alpha)\big) \ , \\
\psi_2 &: ((v, u), \beta) \mapsto \big([v, 1, u], (v\beta, \beta, u\beta)\big) \ , \\
\psi_3 &: ((z, w), \gamma) \mapsto \big([1, z, w], (\gamma, z\gamma, w\gamma)\big) \ .
\end{aligned}$$

For any $\ell \in \mathbb{N} \cup \{0\}$, denote $\otimes_\ell E$ by E^ℓ. This is a commonly used notation when E is a line bundle (a rank 1 vector bundle). Note that the bundle transition function for E^ℓ is the ℓ-th power of that for E.

(a) Find out the coordinate transitions for $\mathbb{RP}^2$, and the coordinate transition functions for E.

(b) Find out the bundle transition functions for $\Lambda^2 T^*\mathbb{RP}^2$, and use it to conclude that $\Lambda^2 T^*\mathbb{RP}^2$ is isomorphic to E^3.

(c) Prove that E^2 is isomorphic to the trivial bundle $\mathbb{RP}^2 \times \mathbb{R}$ by constructing a nowhere vanishing section. [*Hint*: Consider the local section $(x, y) \mapsto 1/(1 + x^2 + y^2)$.]

To sum up, $\Lambda^2 T^*\mathbb{RP}^2 \cong E^3 = (E^2) \otimes E \cong E$. In #1 of Homework 4, you proved that E is not isomorphic to the trivial bundle $\mathbb{RP}^2 \times \mathbb{R}$. Therefore, $\Lambda^2 T^*\mathbb{RP}^2$ does not admit a nowhere vanishing section.

Remark: A manifold M^n is said to be *orientable* if $\Lambda^n T^*M$ is isomorphic to the trivial bundle $M \times \mathbb{R}$. Exercise 1 shows that $\mathbf{S}^2$ is orientable, and Exercise 2 shows that $\mathbb{RP}^2$ is not orientable.

The above method can be used to show that $\Lambda^n T^*\mathbb{RP}^n \cong E^{n+1}$, which is trivial when n is odd and nontrivial when n is even. In other words, $\mathbb{RP}^n$ is orientable if and only if n is odd.

(d) Denote the trivial bundle $\mathbb{RP}^2 \times \mathbb{R}^3$ by $\underline{\mathbb{R}}^3$. The tautological bundle is a subbundle of $\underline{\mathbb{R}}^3$. Find out the bundle transition functions for the quotient bundle $\underline{\mathbb{R}}^3/E$, and prove that $\underline{\mathbb{R}}^3/E$ is isomorphic to the tangent bundle. [*Hint*: Endow $\mathbb{R}^3$ with the standard inner product. Then $E^\perp$ has two linearly independent, local sections: $(1, 0, -x)$ and $(0, 1, -y)$. After a suitable change of basis, you shall be able to prove the isomorphism between $T\mathbb{RP}^2 \otimes E^2$ and $\underline{\mathbb{R}}^3/E$.]

(3) Let Σ be a compact surface[1] in $\mathbb{R}^3$. Consider the standard coordinate functions, x, y, z, for $\mathbb{R}^3$. Since Σ is compact, we may assume that all the points of Σ have positive z-value. Namely, Σ lies above the xy-plane. For any $\mathbf{v} \in \Sigma$, the normalized position vector $\mathbf{v}/|\mathbf{v}|$ defines a smooth map from Σ to $\mathbf{S}^2$. Denote this map by Ψ. Prove that $\Psi^*T\mathbf{S}^2$ is isomorphic to the trivial bundle $\Sigma \times \mathbb{R}^2$. [*Hint*: What is $\Psi^*T\mathbf{S}^2$ as a subbundle of $T\mathbb{R}^3|_\Sigma = \Sigma \times \mathbb{R}^3$? Can you construct an isomorphism to $\Sigma \times \{xy\text{-plane}\} \subset \Sigma \times \mathbb{R}^3$?]

[1] Namely, Σ is a compact, two dimensional submanifold of $\mathbb{R}^3$.