DIFFERENTIAL GEOMETRY I
HOMEWORK 4

DUE: WEDNESDAY, OCTOBER 15

(1) The tautological bundle over $\mathbb{RP}^n$ is defined by

$$E = \big\{([\mathbf{x}], \mathbf{v}) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \bigm| \mathbf{x} \in \mathbf{S}^n, \mathbf{v} \text{ is parallel to } \mathbf{x}\big\} .$$

Prove that E is not isomorphic to the trivial bundle $\mathbb{RP}^n \times \mathbb{R}$. [*Hint*: You may prove it by contradiction. Suppose it is, then there exists a *nowhere vanishing section* $s : \mathbb{RP}^n \to E$. It induces a smooth map

$$\tilde{s} : \mathbf{S}^n \to \mathbb{RP}^n \xrightarrow{s} E \to \mathbb{R}^{n+1}$$

where the first map is the quotient map, and the third map is the restriction of the projection map $\mathbb{RP}^n \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Since $\tilde{s}(\mathbf{x})$ is parallel to $\mathbf{x}$, we can consider the smooth function $f : \mathbf{S}^n \to \mathbb{R}$ defined by $\mathbf{x} \mapsto \tilde{s}(\mathbf{x})/\mathbf{x}$. What can you say about $f(\mathbf{x})$ and $f(-\mathbf{x})$?]

(2) Consider $\mathbf{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ with the following two coordinate charts:

$$\begin{array}{ccc} (0, 2\pi) & \xrightarrow{\varphi_U^{-1}} & U = \{(x, y) \in \mathbf{S}^1 \mid x \neq 1\} \\ \theta_1 & \longmapsto & (\cos\theta_1, \sin\theta_1) \end{array}$$

$$\begin{array}{ccc} (-\pi, \pi) & \xrightarrow{\varphi_V^{-1}} & V = \{(x, y) \in \mathbf{S}^1 \mid x \neq -1\} \\ \theta_2 & \longmapsto & (\cos\theta_2, \sin\theta_2) \end{array}$$

(a) Show that $T^*\mathbf{S}^1$ is isomorphic to the trivial bundle $\mathbf{S}^1 \times \mathbb{R}$. [*Hint*: You can work out the transition function, and consider the coordinate differential on these two charts.]

(b) Construct a 1-form on $\mathbf{S}^1$ which is not the differential of a smooth function. [*Remark*: The 1-forms $\{\mathrm{d}f \mid f \in \mathcal{C}^\infty(\mathbf{S}^1; \mathbb{R})\}$ are called *exact* 1-forms.]

(c) Show that any 1-form on $\mathbb{R}^1$, $g(x)\mathrm{d}x$, is always the differential of some smooth function $f(x)$ on $\mathbb{R}^1$.

From Part (b) and (c), you can see that functions and 1-forms capture certain topological information of the manifold.

(3) The main purpose of this exercise is to redo #2 of Homework 1. Sometimes it is easier to work with the "ambient coordinate" than with the "intrinsic coordinate".

(a) Let M be a submanifold of $\mathbb{R}^N$ whose dimension is strictly less than N. Suppose that $F : \mathbb{R}^N \to \mathbb{R}$ is a smooth function, and denote by f the restriction of F on M. There are two ways to associate a 1-form on M.

(i) Take the differential of f: $\mathrm{d}f$.

(ii) The differential of F, $\mathrm{d}F$, is a 1-form on $\mathbb{R}^N$, which is a smooth function on $T\mathbb{R}^N$ and is fiberwise linear[1]. As explained in class, TM is naturally a submanifold of $T\mathbb{R}^N = \mathbb{R}^N \times \mathbb{R}^N$. Consider the restriction of $\mathrm{d}F$ on TM, which is usually denoted by $(\mathrm{d}F)|_M$. It is not hard to see that $(\mathrm{d}F)|_M$ is smooth and fiberwise linear, and thus is a 1-form on M.

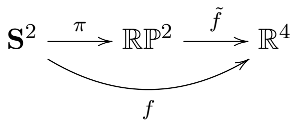
Prove that $\mathrm{d}f = (\mathrm{d}F)|_M$. [*Hint*: You may use the (non-linear local) coordinate introduced by discussion 2.2.]

(b) Recall the map in #2 of Homework 1, Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$F(x, y, z) = (x^2 - y^2, xy, zx, yz) \ .$$

Denote the restriction of F on $\mathbf{S}^2$ by f. Prove that f is an immersion. [*Hint*: Part (a) says that you can use $(\mathrm{d}F)|_{\mathbf{S}^2}$. When $z \neq \pm 1$, consider the following two vector fields (on $\mathbf{S}^2 \backslash \{N, S\}$): $\mathbf{v}_1 = (-y, x, 0), \mathbf{v}_2 = \mathbf{x} \times \mathbf{v}_1 = (-xz, -yz, x^2 + y^2)$. They form a basis of $T_{\mathbf{x}}\mathbf{S}^2$ for any $\mathbf{x} \in \mathbf{S}^2 \backslash \{N, S\}$.]

Denote the induced map on $\mathbb{RP}^2$ by $\tilde{f}$. It follows that $\tilde{f}$ is an immersion. (Think about it. You don't have to submit the $\tilde{f}$ part.)



Remark: You can apply the same argument to prove that $\mathbb{RP}^3$ is diffeomorphic to $\mathrm{SO}(3)$ (#2 of Homework 2). The vector fields in #2.d of Homework 3 will be useful.

(4) Consider $\mathbf{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with the stereographic projection:

$$\varphi_U^{-1} : \mathbb{R}^2 \to U = \mathbf{S}^2 \backslash \{(0,0,1)\} \qquad \varphi_V^{-1} : \mathbb{R}^2 \to V = \mathbf{S}^2 \backslash \{(0,0,-1)\}$$

$$\mathbf{u} \mapsto \left(\frac{2\mathbf{u}}{1 + |\mathbf{u}|^2}, \frac{-1 + |\mathbf{u}|^2}{1 + |\mathbf{u}|^2}\right) \qquad \mathbf{v} \mapsto \left(\frac{2\mathbf{v}}{1 + |\mathbf{v}|^2}, \frac{1 - |\mathbf{v}|^2}{1 + |\mathbf{v}|^2}\right)$$

The coordinate transition $\varphi_V \circ \varphi_U^{-1}$ sends $\mathbf{u}$ to $\mathbf{v} = \mathbf{u}/|\mathbf{u}|^2$. Write down the following two vector fields in terms of (V, φ_V):

(a) $u^1 \frac{\partial}{\partial u^1} + u^2 \frac{\partial}{\partial u^2}$;

(b) $-u^2 \frac{\partial}{\partial u^1} + u^1 \frac{\partial}{\partial u^2}$.

[1] Namely, $\mathrm{d}F|_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^N = \{\mathbf{x}\} \times \mathbb{R}^N \to \mathbb{R}$ is linear for any $\mathbf{x} \in \mathbb{R}^N$