DIFFERENTIAL GEOMETRY I
HOMEWORK 3

DUE: WEDNESDAY, OCTOBER 8

(1) Suppose that $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map with the property that

$$\psi(\lambda\mathbf{x}) = \lambda\psi(\mathbf{x}) \quad \text{for any } \lambda \in \mathbb{R} \text{ and } \mathbf{x} \in \mathbb{R}^n \ .$$

It is clear that $\psi(\mathbf{0})$ must be $\mathbf{0}$.

(a) When $n = 1$, show that ψ is a linear function. [*Hint*: The derivative $\psi'(\mathbf{x})$ is a constant.]

(b) When $n \geq 2$, prove that ψ is a linear map. [*Hint*: Compare ψ with its linearization at the origin.]

(2) Consider the matrix group

$$\mathrm{SU}(n) = \big\{\mathfrak{m} \in \mathrm{Gl}(n;\mathbb{C}) \ \big|\ \mathfrak{m}\mathfrak{m}^* = \mathbf{I} \text{ and } \det(\mathfrak{m}) = 1\big\} \ .$$

(a) Prove that $\mathrm{SU}(n)$ is a Lie group by showing that

$$\begin{aligned} \psi : \quad \mathrm{Gl}(n;\mathbb{C}) \quad &\to \quad \mathrm{Herm}(n) \times \mathbb{R} \\ \mathfrak{m} \quad &\mapsto \quad \big(\mathfrak{m}\mathfrak{m}^* - \mathbf{I}, \tfrac{-i}{2}(\det(\mathfrak{m}) - \det(\mathfrak{m}^*))\big) \end{aligned}$$

has $(\mathbf{0}, 0)$ as its regular value. Here, $\mathrm{Herm}(n)$ is the set of all $n \times n$ Hermitian matrices, which is isomorphic to $\mathbb{R}^{n^2}$ as a real vector space. The manifold $\psi^{-1}(\mathbf{0}, 0)$ has two components, and $\mathrm{SU}(n)$ is the component containing the identity matrix. It follows that the (real) dimension of $\mathrm{SU}(n)$ is $n^2 - 1$.

(b) Describe the tangent bundle of $\mathrm{SU}(n)$ as a subset of $\mathbb{M}(n;\mathbb{C}) \times \mathbb{M}(n;\mathbb{C})$.

(c) Focus on the case when $n = 2$. Show that $\mathrm{SU}(2)$ is the same as $\mathbf{S}^3$. Although they are in fact diffeomorphic, you are only asked to argue it set-theoretically. [*Hint*: For any $\mathfrak{m} \in \mathrm{SU}(2)$, what can you say about its first column vector? After fixing the first column, how many choices do you have for the second column?]

(d) The tangent space of $\mathbf{S}^3$ can be described by

$$T\mathbf{S}^3 = \big\{(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^4 \times \mathbb{R}^4 \ \big|\ |\mathbf{x}| = 1 \text{ and } \mathbf{v} \perp \mathbf{x}\big\} \ .$$

Write down three (smooth) vector fields on $\mathbf{S}^3$ that are linearly independent at every $\mathbf{x} \in \mathbf{S}^3$. [*Hint*: You can get some idea from Part (b) and (c).]

(3) Consider $\mathbf{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with the stereographic projection:

$$\begin{aligned} \varphi_U^{-1} : \mathbb{R}^2 &\to U = \mathbf{S}^2 \backslash \{(0, 0, 1)\} \\ \mathbf{u} &\mapsto \Big(\frac{2\mathbf{u}}{1 + |\mathbf{u}|^2}, \frac{-1 + |\mathbf{u}|^2}{1 + |\mathbf{u}|^2}\Big) \end{aligned} \qquad \begin{aligned} \varphi_V^{-1} : \mathbb{R}^2 &\to V = \mathbf{S}^2 \backslash \{(0, 0, -1)\} \\ \mathbf{v} &\mapsto \Big(\frac{2\mathbf{v}}{1 + |\mathbf{v}|^2}, \frac{1 - |\mathbf{v}|^2}{1 + |\mathbf{v}|^2}\Big) \end{aligned}$$

(a) Write down the bundle transition function $g_{U,V}$ for the tangent bundle $T\mathbf{S}^2$.

(b) Write down the bundle transition function $g_{U,V}$ for the cotangent bundle $T^*\mathbf{S}^2$.

(c) Consider the restriction of the function $(x, y, z) \mapsto z^2$ on $\mathbf{S}^2$. Denote it by f. Write down its differential $\mathrm{d}f$ using the coordinate charts (U, φ_U) and (V, φ_V). Check that your expression obeys the bundle transition function of Part (b).

Here are two points about this computation:

- $\mathrm{d}f$ does define a section of the cotangent bundle of $\mathbf{S}^2$, which is called a 1-form on $\mathbf{S}^2$;
- this is a double-check of your result of Part (b).